1. Natural Numbers

1 The Set of Natural Numbers

Definition 1.1. The set of natural numbers $\mathbb{N}$ is defined by the following axioms:

Axiom 1 (Peano Axioms)**.** The natural numbers satisfy:

1. $1 \in \mathbb{N}$ (existence of a unique initial element)
2. For each $n \in \mathbb{N}$, there exists a unique successor $n + 1 \in \mathbb{N}$
3. 1 is not the successor of any natural number
4. If $n, m \in \mathbb{N}$ have the same successor, then $n = m$ (injectivity of successor)
5. If $S \subseteq \mathbb{N}$ contains 1 and contains $n+1$ whenever it contains n, then $S = \mathbb{N}$ (principle of induction)

2 Mathematical Induction

Theorem 2.1 (Principle of Mathematical Induction)**.** *Let $P_1, P_2, \ldots$ be a sequence of mathematical statements such that:*

1. *P_1 is true (base case)*
2. *For each $n \in \mathbb{N}$, if P_n is true then P_{n+1} is true (inductive step)*

Then P_n is true for all $n \in \mathbb{N}$.

Proof. Let $S = \{n \in \mathbb{N} : P_n \text{ is true}\}$. By assumption (1), $1 \in S$. By assumption (2), if $n \in S$ then $n + 1 \in S$. By Axiom 5, $S = \mathbb{N}$. □

Example 2.2 (Sum of First n Natural Numbers)**.** Prove that $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$ for all $n \in \mathbb{N}$.

Proof. Let P_n be the statement: $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$.
Base case ($n = 1$): $1 = \frac{1}{2} \cdot 1 \cdot 2 = 1$, so P_1 is true.
Inductive step: Assume P_n is true. Then:

$$\begin{aligned} 1 + 2 + \cdots + n + (n + 1) &= \frac{1}{2}n(n + 1) + (n + 1) \\ &= (n + 1)\left(\frac{n}{2} + 1\right) \\ &= \frac{1}{2}(n + 1)(n + 2) \end{aligned}$$

Thus P_{n+1} is true. By mathematical induction, P_n holds for all $n \in \mathbb{N}$. □

Example 2.3 (Divisibility Property)**.** Prove that $5^n - 4n - 1$ is divisible by 16 for all $n \in \mathbb{N}$.

Proof. **Base case ($n = 1$):** $5 - 4 - 1 = 0$, which is divisible by 16.
Inductive step: Assume $5^n - 4n - 1 = 16k$ for some integer k. Then:

$$\begin{aligned} 5^{n+1} - 4(n + 1) - 1 &= 5 \cdot 5^n - 4n - 4 - 1 \\ &= 5(5^n - 4n - 1) + 16n \\ &= 5 \cdot 16k + 16n = 16(5k + n) \end{aligned}$$

which is divisible by 16. Thus the statement holds for all $n \in \mathbb{N}$. □

Example 2.4 (Trigonometric Inequality)**.** Prove that $|\sin(nx)| \leq n|\sin x|$ for all positive integers n and real numbers x.

Homework Problems

1, 4, 8, 11, 12