1 Monotone Sequences

Definition 1. *A sequence* (s_n) *is called:*

- ***increasing*** *if* $s_n \leq s_{n+1}$ *for all* $n \in \mathbb{N}$*,*
- ***decreasing*** *if* $s_n \geq s_{n+1}$ *for all* $n \in \mathbb{N}$*.*

Increasing and decreasing sequences are collectively called ***monotone*** *sequences.*

Theorem 1 (Monotone Convergence Theorem)**.** *Every bounded monotone sequence converges.*

Proof. Let (s_n) be a bounded increasing sequence. Let $S = \{s_n : n \in \mathbb{N}\}$ and $u = \sup S \in \mathbb{R}$. We claim $\lim_{n\to\infty} s_n = u$.

For any $\epsilon > 0$, since $u - \epsilon$ is not an upper bound for S, there exists N such that $s_N > u - \epsilon$. Since (s_n) is increasing, for all $n \geq N$ we have $s_N \leq s_n \leq u$. Therefore, $|s_n - u| < \epsilon$ for all $n \geq N$.

The proof for decreasing sequences is analogous. □

Example 1. *Consider the sequence defined by:*

$$s_1 = \sqrt{2}, \quad s_n = \frac{s_{n-1}^2 + 2}{2s_{n-1}} \quad \text{for } n \geq 2.$$

By mathematical induction, we can show that $s_n > s_{n+1}$ *for all* n*, so the sequence is decreasing and bounded below. Hence, the limit exists. Let* $\lim_{n\to\infty} s_n = s$*. Then:*

$$2s \cdot s = s^2 + 2 \quad \Rightarrow \quad s^2 = 2 \quad \Rightarrow \quad s = \sqrt{2}.$$

2 Decimal Expansions

Consider a decimal number of the form:

$$k.d_1 d_2 d_3 \ldots$$

where $k \geq 0$ is an integer and $0 \leq d_i \leq 9$ are digits. Define the sequence:

$$s_n = k + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}.$$

Then (s_n) is increasing and bounded above by $k+1$. Therefore, (s_n) converges to some real number a.

Example 2. *The sequence* $s_n = 0.\underbrace{99\ldots9}_{n \text{ times}} = 1 - \frac{1}{10^n}$ *converges to 1.*

Theorem 2. *If* (s_n) *is unbounded and increasing, then* $\lim_{n\to\infty} s_n = +\infty$*. Similarly, if* (s_n) *is unbounded and decreasing, then* $\lim_{n\to\infty} s_n = -\infty$*.*

Proof. Suppose (s_n) is unbounded and increasing. Then for any $M > 0$, there exists N such that $s_N > M$. Since (s_n) is increasing, $s_n \geq s_N > M$ for all $n \geq N$. Hence, $\lim_{n\to\infty} s_n = +\infty$. □

3 Limit Superior and Limit Inferior

Definition 2. *For a sequence (s_n), define:*

$$a_N = \sup\{s_n : n \geq N\},$$
$$b_N = \inf\{s_n : n \geq N\}.$$

*Then the **limit superior** and **limit inferior** are defined as:*

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} a_N,$$
$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} b_N.$$

Note that (a_N) is decreasing and (b_N) is increasing, so these limits always exist (possibly $\pm\infty$).

Theorem 3. *Let (s_n) be a sequence.*

1. *If $\lim_{n\to\infty} s_n$ exists (finite or infinite), then:*

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

2. *If $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$ (finite or infinite), then $\lim_{n\to\infty} s_n$ exists and equals this common value.*

Proof. (1) Suppose $\lim_{n\to\infty} s_n = s \in \mathbb{R}$. For any $\epsilon > 0$, there exists N such that $|s_n - s| < \epsilon$ for all $n \geq N$. Then:

$$s - \epsilon \leq b_N \leq s_n \leq a_N \leq s + \epsilon.$$

Taking limits as $N \to \infty$, we get:

$$s - \epsilon \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq s + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the result follows.

(2) Suppose $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = L$. Then for any $\epsilon > 0$, there exists N such that for all $n \geq N$:

$$L - \epsilon \leq b_N \leq s_n \leq a_N \leq L + \epsilon.$$

Hence, $|s_n - L| < \epsilon$ for all $n \geq N$, so $\lim_{n\to\infty} s_n = L$. □

4 Cauchy Sequences

Definition 3. *A sequence (s_n) is called a **Cauchy sequence** if for every $\epsilon > 0$, there exists N such that for all $m, n \geq N$:*

$$|s_n - s_m| < \epsilon.$$

Theorem 4. *Every convergent sequence is Cauchy.*

Proof. Suppose $\lim_{n\to\infty} s_n = s$. For any $\epsilon > 0$, there exists N such that $|s_n - s| < \epsilon/2$ for all $n \geq N$. Then for all $m, n \geq N$:

$$|s_n - s_m| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Lemma 5. *Every Cauchy sequence is bounded.*

Proof. Let $\epsilon = 1$. There exists N such that $|s_n - s_m| < 1$ for all $m, n \geq N$. In particular, for $n \geq N$:

$$|s_n| \leq |s_n - s_N| + |s_N| < 1 + |s_N|.$$

Let $M = \max\{|s_1|, \ldots, |s_{N-1}|, 1 + |s_N|\}$. Then $|s_n| \leq M$ for all n. □

Theorem 6 (Cauchy Criterion)**.** *A sequence (s_n) converges if and only if it is Cauchy.*

Proof. $(\Rightarrow)$ Already proved.

$(\Leftarrow)$ Suppose (s_n) is Cauchy. Then it is bounded. Let:

$$a = \limsup_{n\to\infty} s_n, \quad b = \liminf_{n\to\infty} s_n.$$

We will show $a = b$. For any $\epsilon > 0$, there exists N such that $|s_n - s_m| < \epsilon/2$ for all $m, n \geq N$. Fix $m \geq N$. Then:

$$s_m - \frac{\epsilon}{2} \leq s_n \leq s_m + \frac{\epsilon}{2} \quad \text{for all } n \geq N.$$

Taking supremum and infimum over $n \geq N$, we get:

$$s_m - \frac{\epsilon}{2} \leq b_N \leq a_N \leq s_m + \frac{\epsilon}{2}.$$

Then:

$$0 \leq a - b \leq a_N - b_N \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $a = b$. Hence, $\lim_{n\to\infty} s_n$ exists. □