1 Subsequences

Definition 1. *Let (s_n) be a sequence and let (n_k) be a strictly increasing sequence of natural numbers. Then the sequence (s_{n_k}) is called a* ***subsequence*** *of (s_n).*

Example 1. *Let $(s_n) = (\frac{1}{n}) = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. If $(n_k) = 1, 3, 8, 10, \ldots$, then the corresponding subsequence is $(s_{n_k}) = 1, \frac{1}{3}, \frac{1}{8}, \frac{1}{10}, \ldots$.*

2 Subsequential Limits

Theorem 1. *Let (s_n) be a sequence and $t \in \mathbb{R}$.*

1. *There exists a subsequence of (s_n) converging to t if and only if for every $\epsilon > 0$, the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite.*
2. *If (s_n) is unbounded above, then it has a subsequence converging to $+\infty$.*
3. *If (s_n) is unbounded below, then it has a subsequence converging to $-\infty$.*

Proof. (1) ($\Leftarrow$) Suppose for every $\epsilon > 0$, the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite. We construct a subsequence converging to t as follows:

Let $\epsilon_1 = 1$. Since $\Omega_1 = \{n \in \mathbb{N} : |s_n - t| < 1\}$ is infinite, pick $n_1 \in \Omega_1$.

Let $\epsilon_2 = \frac{1}{2}$. Since $\Omega_2 = \{n \in \mathbb{N} : |s_n - t| < \frac{1}{2}\}$ is infinite, pick $n_2 \in \Omega_2$ with $n_2 > n_1$.

Continue this process: for each $k \in \mathbb{N}$, let $\epsilon_k = \frac{1}{k}$. Since $\Omega_k = \{n \in \mathbb{N} : |s_n - t| < \frac{1}{k}\}$ is infinite, pick $n_k \in \Omega_k$ with $n_k > n_{k-1}$.

This gives a strictly increasing sequence (n_k) such that $|s_{n_k} - t| < \frac{1}{k}$ for all k. Hence, $\lim_{k\to\infty} s_{n_k} = t$.

($\Rightarrow$) If a subsequence converges to t, then for any $\epsilon > 0$, infinitely many terms satisfy $|s_{n_k} - t| < \epsilon$. □

3 Existence of Monotone Subsequences

Theorem 2. *Every sequence has a monotone subsequence.*

Proof. A term s_n is called **dominant** if $s_n > s_m$ for all $m > n$.

Case 1: There are infinitely many dominant terms. Then these terms form a strictly decreasing subsequence.

Case 2: There are only finitely many dominant terms. Pick n_1 beyond all dominant terms. Since s_{n_1} is not dominant, there exists $n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$. Since s_{n_2} is not dominant, there exists $n_3 > n_2$ such that $s_{n_3} \geq s_{n_2}$. Continuing this process, we obtain an increasing subsequence. □

Theorem 3 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Proof. By the previous theorem, every sequence has a monotone subsequence. If the original sequence is bounded, then this monotone subsequence is also bounded. By the Monotone Convergence Theorem, it converges. □

4 Subsequential Limits and Limit Superior/Inferior

Definition 2. *A **subsequential limit** of (s_n) is the limit of any convergent subsequence of (s_n).*

Example 2. *For the sequence $s_n = (-1)^n$, the subsequential limits are 1 and -1.*

Theorem 4. *For any sequence (s_n), there exists a monotone subsequence whose limit is $\limsup_{n\to\infty} s_n$. Similarly, there exists a monotone subsequence whose limit is $\liminf_{n\to\infty} s_n$.*

Proof. Let $t = \limsup_{n\to\infty} s_n$ and $a_N = \sup\{s_n : n \geq N\}$. Then $\lim_{N\to\infty} a_N = t$.

We show that for every $\epsilon > 0$, the set $\{n : |s_n - t| < \epsilon\}$ is infinite. Suppose not. Then there exists $\epsilon > 0$ such that only finitely many s_n satisfy $|s_n - t| < \epsilon$. But then for large N, $a_N \leq t - \epsilon$ or $a_N \geq t + \epsilon$, contradicting $\lim_{N\to\infty} a_N = t$.

Now construct a subsequence converging to t as in the proof of Theorem 2.1. □

Theorem 5. *Let S be the set of all subsequential limits of (s_n). Then:*

$$\sup S = \limsup_{n\to\infty} s_n \quad \textit{and} \quad \inf S = \liminf_{n\to\infty} s_n.$$

Proof. We show $\sup S = \limsup_{n\to\infty} s_n$. Let $t = \limsup_{n\to\infty} s_n$. By the previous theorem, $t \in S$, so $t \leq \sup S$.

Now let $s \in S$. Then there exists a subsequence (s_{n_k}) converging to s. For each k, $s_{n_k} \leq a_{n_k} = \sup\{s_n : n \geq n_k\}$. Taking limits:

$$s = \lim_{k\to\infty} s_{n_k} \leq \lim_{k\to\infty} a_{n_k} = t.$$

Hence, $\sup S \leq t$. Therefore, $\sup S = t$.

The proof for $\inf S = \liminf_{n\to\infty} s_n$ is similar. □

Theorem 6. *The set S of subsequential limits is closed.*

Proof. Let (t_n) be a sequence in $S \cap \mathbb{R}$ converging to t. We show $t \in S$.

For any $\epsilon > 0$, there exists $t_n \in (t - \epsilon, t + \epsilon)$. Choose $\delta > 0$ such that $(t_n - \delta, t_n + \delta) \subseteq (t - \epsilon, t + \epsilon)$. Since t_n is a subsequential limit, the set $\{m : s_m \in (t_n - \delta, t_n + \delta)\}$ is infinite. Hence, $\{m : s_m \in (t - \epsilon, t + \epsilon)\}$ is also infinite. Therefore, we can construct a subsequence converging to t. □