Limit Superior

1 Limit Superior and Products of Sequences

Theorem 1. *Suppose* $\lim_{n\to\infty} S_n = S$*, where* $0 < S < \infty$*. Then*

$$\limsup_{n\to\infty}(S_n t_n) = S \cdot \limsup_{n\to\infty} t_n.$$

Proof. We show both inequalities.

Let $\beta = \limsup_{n\to\infty} t_n$.

Case 1: β is finite. There exists a subsequence t_{n_k} such that $t_{n_k} \to \beta$. Since $S_{n_k} \to S$, we have

$$S_{n_k} t_{n_k} \to S \cdot \beta.$$

Therefore,

$$S \cdot \beta \leq \limsup_{n\to\infty}(S_n t_n).$$

Case 2: $\beta = +\infty$. There exists a subsequence $t_{n_k} \to +\infty$. Since $S_{n_k} \to S > 0$, we have

$$S_{n_k} t_{n_k} \to +\infty,$$

so the inequality holds.

Case 3: $\beta = -\infty$. Then $S \cdot \beta = -\infty$, and the inequality is trivial.

Now we prove the reverse inequality. Ignoring finitely many terms, assume $S_n \neq 0$ for all n. Then

$$\lim_{n\to\infty} \frac{1}{S_n} = \frac{1}{S}.$$

Replacing S_n by $\frac{1}{S_n}$ and t_n by $S_n t_n$ in the previous result, we get

$$\limsup_{n\to\infty} t_n = \limsup_{n\to\infty} \left(\frac{1}{S_n} \cdot (S_n t_n) \right) \geq \frac{1}{S} \cdot \limsup_{n\to\infty}(S_n t_n).$$

Multiplying both sides by S gives

$$\limsup_{n\to\infty}(S_n t_n) \leq S \cdot \limsup_{n\to\infty} t_n.$$

This completes the proof. □

2 Root Test and Ratio Test

Theorem 2. *Let $\{s_n\}$ be a sequence of nonzero real numbers. Define*

$$L = \limsup_{n\to\infty} \left| \frac{s_{n+1}}{s_n} \right|.$$

Then

$$\limsup_{n\to\infty} |s_n|^{1/n} \le L.$$

Moreover, if $\lim_{n\to\infty} \left| \frac{s_{n+1}}{s_n} \right|$ *exists and equals L, then*

$$\lim_{n\to\infty} |s_n|^{1/n} = L.$$

Proof. Let $\alpha = \limsup_{n\to\infty} |s_n|^{1/n}$. We want to show $\alpha \le L$.

If $L = +\infty$, the result is obvious. Assume $L < \infty$. Pick any $L_1 > L$. Then there exists N such that

$$\sup \left\{ \left| \frac{s_{n+1}}{s_n} \right| : n \ge N \right\} < L_1,$$

so

$$\left| \frac{s_{n+1}}{s_n} \right| < L_1 \quad \text{for all } n \ge N.$$

For $n > N$, we have

$$|s_n| = \left| \frac{s_n}{s_{n-1}} \right| \cdot \left| \frac{s_{n-1}}{s_{n-2}} \right| \cdots \left| \frac{s_{N+1}}{s_N} \right| \cdot |s_N| < L_1^{n-N} |s_N|.$$

Taking the n-th root:

$$|s_n|^{1/n} < L_1^{1-N/n} |s_N|^{1/n}.$$

Taking limit superior:

$$\alpha \le \limsup_{n\to\infty} L_1^{1-N/n} |s_N|^{1/n} = L_1.$$

Since $L_1 > L$ was arbitrary, we conclude $\alpha \le L$.

If the limit of the ratios exists and equals L, then a similar argument shows that $\lim_{n\to\infty} |s_n|^{1/n} = L$. □