13. Metric Spaces and Topology

We work on $\mathbb{R}$ as an ordered field. However, in $\mathbb{R}^n$ (for $n > 1$), there is no natural ordering, so we cannot use the absolute value directly. Instead, we base our analysis on the concept of distance.

Definition 1. *Let S be a set. A function $d : S \times S \to \mathbb{R}$ is called a* ***distance function*** *or a* ***metric*** *if it satisfies:*

D1. $d(x, x) = 0$ *for all* $x \in S$*, and* $d(x, y) > 0$ *for* $x \neq y$

D2. $d(x, y) = d(y, x)$ *for all* $x, y \in S$ *(symmetry)*

D3. $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in S$ *(triangle inequality)*

The pair (S, d) is called a ***metric space****.*

Example 1. *On $\mathbb{R}$, $d(a, b) = |a - b|$ is a metric.*

Example 2. *On $\mathbb{R}^n$, for $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$, define:*

$$d(\vec{x}, \vec{y}) = \left[\sum_{j=1}^{n} (x_j - y_j)^2 \right]^{\frac{1}{2}}$$

This is the Euclidean metric on $\mathbb{R}^n$.

Topology

A **topology** on a set S is a collection σ of subsets of S such that:

1. $\emptyset \in \sigma$ and $S \in \sigma$
2. The union of any subcollection of elements in σ is in σ
3. The intersection of any finitely many elements in σ is in σ

Elements in σ are called **open sets** of S. A subset $U \subset S$ is called **closed** if its complement is open.

Example 3. *Let $S = \{a, b, c\}$. Then:*

$$\sigma = \{\emptyset, S, \{a, b\}\}$$

is a topology on S.

We are particularly interested in topologies induced by metrics.

Definition 2. *Let (S, d) be a metric space. A subset $U \subset S$ is said to be* ***open*** *if for every $s \in U$, there exists $\delta > 0$ such that the open ball:*

$$B(s, \delta) = \{s' \in S \mid d(s, s') < \delta\}$$

is contained in U.

The collection of all such open sets forms a topology, called the **metric topology**.

Example 4. *In $(\mathbb{R}, d)$ with $d(a, b) = |a - b|$:*

- $(1, 2)$ *is open*
- $(-\infty, 1)$ *and* $(2, \infty)$ *are open*
- $[1, 2]$ *is closed*
- *Open sets are unions of open intervals*

Convergence and Completeness

Definition 3. *A sequence (ξ_n) in a topological space (ζ, σ)* ***converges*** *to ξ if for any open set $U \in \sigma$ containing ξ, there exists N such that for all $n > N$, $\xi_n \in U$.*

Lemma 1. *Suppose the topology is induced by a metric d. A sequence (s_n) in ζ converges to s if and only if:*

$$\lim_{n \to \infty} d(s_n, s) = 0$$

Definition 4. *A sequence (s_n) in a metric space (ζ, d) is a* ***Cauchy sequence*** *if for every $\epsilon > 0$, there exists N such that for all $m, n > N$, $d(s_m, s_n) < \epsilon$.*

Definition 5. *A metric space (S, d) is* ***complete*** *if every Cauchy sequence in S converges to a point in S.*

Completeness of $\mathbb{R}^k$

Our goal is to show that $(\mathbb{R}^k, d^2)$ is complete, where d^2 is the Euclidean metric.

We use superscripts to denote sequences: $(\vec{x}^n)$, where $\vec{x}^n = (x_1^n, x_2^n, \dots, x_k^n)$.

Lemma 2. *A sequence $(\vec{x}^n)$ in $\mathbb{R}^k$ converges if and only if for each $j = 1, \dots, k$, the sequence (x_j^n) converges as $n \to \infty$.*

Lemma 3. *A sequence $(\vec{x}^n)$ in $\mathbb{R}^k$ is a Cauchy sequence if and only if for each $j = 1, \dots, k$, the sequence (x_j^n) is a Cauchy sequence in $\mathbb{R}$.*

Theorem 1. $\mathbb{R}^k$ *(with respect to the Euclidean distance) is complete.*

Proof. Consider a Cauchy sequence $(\vec{x}^n)$ in $\mathbb{R}^k$. Then for each $j = 1, \dots, k$, the sequence (x_j^n) is a Cauchy sequence in $\mathbb{R}$, hence converges to some x_j. Then $(\vec{x}^n)$ converges to $(x_1, \dots, x_k)$. □

Bounded Sets and Bolzano-Weierstrass Theorem

Definition 6. *A set S in $\mathbb{R}^n$ is **bounded** if there exists $M > 0$ such that:*

$$\max\{|x_j| : j = 1, \dots, k\} \leq M \quad \textit{for all } \vec{x} \in S$$

Theorem 2 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.*

Proof. Let (x^n) be a bounded sequence in $\mathbb{R}^k$. Then for each j, the sequence (x_j^n) is bounded. By the one-dimensional Bolzano-Weierstrass theorem, we can select a subsequence of (x^n), still denoted by (x^n), such that (x_1^n) converges. Repeating this process for each coordinate, we obtain a subsequence where (x_j^n) converges for each $j = 1, \dots, k$. Then (x^n) converges. □

Properties of Closed Sets

Proposition 1. *Let $(S, \mathcal{J})$ be a topological space with topology induced by a metric d on S. Let $E \subset S$.*

(a) E is closed if and only if $E = \overline{E}$ (where $\overline{E}$ is the closure of E, defined as the intersection of all closed subsets containing E)

(b) *E is closed if and only if it contains the limit of every convergent sequence of points in E*

(c) *$s \in \overline{E}$ if and only if s is the limit of some sequence of points in E*

Theorem 3. *Let (F_n) be a decreasing sequence of closed, bounded, non-empty sets in $\mathbb{R}^k$. Then $F = \cap_{n=1}^{\infty} F_n$ is also closed, bounded, and non-empty.*

Proof. We have:

$$F = \cap_{n=1}^{\infty} F_n = (\cup_{n=1}^{\infty} F_n^c)^c$$

Since each F_n^c is open, their union is open, and thus F is closed as the complement of an open set. F is clearly bounded.

To show F is non-empty: For each n, select $x_n \in F_n$. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_{n_m}) converging to x_0. For any n_0, if m is large enough so that $n_m > n_0$, then $x_{n_m} \in F_{n_0}$. Since F_{n_0} is closed, $x_0 \in F_{n_0}$ by Proposition (b). Hence $x_0 \in F$. □

Cantor Set

The **Cantor set** F is constructed as follows:

- $F_1 = [0, 1]$
- $F_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
- $F_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$
- Continue this process, where F_{n+1} is obtained by removing the open middle third of each interval in F_n

The Cantor set $F = \cap_{n=1}^{\infty} F_n$ has the following properties:

- The length of F_{n+1} is $\frac{2}{3}$ times the length of F_n, so the length of F_n is $(\frac{2}{3})^n \to 0$
- F has measure zero
- F does not contain any open interval
- F is uncountable

We can construct a surjective map $\Phi : F \to [0, 1]$ by taking ternary expansions with digits 0 and 2 and converting them to binary expansions. If F were countable, then $[0, 1]$ would be countable, which is a contradiction.

Compactness

Definition 7. *Let $(S, \mathcal{J})$ be a topological space and $E \subset S$. A collection $\mathcal{U}$ of open sets is called an* ***open cover*** *of E if $E \subset \bigcup_{U \in \mathcal{U}} U$.*

A ***subcover*** *of $\mathcal{U}$ is a subcollection $\mathcal{U}' \subset \mathcal{U}$ that is still an open cover of E.*

We say $E \subset S$ is ***compact*** *if every open cover of E has a finite subcover.*

Theorem 4 (Heine-Borel Theorem)**.** *A subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.*