14. Series

Summation notation

$\sum_{k=m}^{n} a_k$ is short hand for

$$a_m + a_{m+1} + \cdots + a_n .$$

Example $\sum_{k=2}^{5} \frac{1}{k^2+k}$

$$= \frac{1}{2^2+2} + \frac{1}{3^2+3} + \frac{1}{4^2+4} + \frac{1}{5^2+5}$$

The symbol $\sum_{n=m}^{\infty} a_n$ is short hand for

$$a_m + a_{m+1} + a_{m+2} + \cdots .$$

But we need to be precise what the meaning.

To assign meaning to $\sum_{n=m}^{\infty} a_n$ we consider the seq $(s_n)_{n=m}^{\infty}$ of partial sums

$$s_n = a_m + a_{m+1} + \cdots + a_n = \sum_{k=m}^{n} a_k$$

The infinite series

$\sum_{n=m}^{\infty} a_n$ is said to converge provided the seq of partial sums converges to a real number S, write $\sum_{n=m}^{\infty} a_n = S$

$$\iff \lim_{n\to\infty} S_n = S$$

$$\iff \lim_{n\to\infty} \left(\sum_{k=m}^{n} a_k \right) = S.$$

A series does not converge is said to diverge

We say $\sum_{n=m}^{\infty} a_n$ diverges to $+\infty / -\infty$
if $\lim S_n = +\infty / -\infty$.

($\sum_{n=m}^{\infty} a_n$ has no meaning if does not converge or diverge to ∞)

Sometimes we do care the exact value
but only whether it converges or diverges
we simply write $\sum a_n$

If $a_n \geq 0$ then S_n is increasing
then either S_n converges
or $\lim S_n = +\infty$

If $a_n \leq 0$, ...

Ex. (geometric series)

$\sum_{n=0}^{\infty} ar^n$ $\quad a, r$ constants.

If $r \neq 1$, $\displaystyle\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$

Pf.

$$(1-r)\sum_{k=0}^{n} ar^k = \sum_{k=0}^{n} ar^k - \sum_{k=0}^{n} ar^{k+1}$$

$$= a + ar + \cdots + ar^n$$
$$- (ar + ar^2 + \cdots + ar^n + ar^{n+1})$$
$$= a - ar^{n+1}$$

$$\Rightarrow \sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}.$$

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

in particular for $p \leq 1$. $\displaystyle\sum \frac{1}{n^p} = \infty$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = ?$ some nice formula

Def. We say $\sum a_n$ satisfies the Cauchy criterion if its seq (s_n) of partial sums is a

Cauchy seq:

for each $\varepsilon > 0$, $\exists N$ s.t.

$$m, n > N \Rightarrow |s_n - s_m| < \varepsilon.$$

$\Leftrightarrow$ for each $\varepsilon > 0$, $\exists N$ s.t.

$$n \geq m > N \Rightarrow |s_n - s_{m-1}| < \varepsilon$$

$$\Big\| \qquad\qquad$$

$$\left| \sum_{k=m}^{n} a_k \right| \quad \star$$

Thm. A series converges iff it satisfies Cauchy criterion.

Cor: $\sum a_n$ converges $\Rightarrow \lim a_n = 0$

from $\star$ let $m = n$

$$|a_n| < \varepsilon \quad \text{for all } n > N.$$

i.e. $\lim a_n = 0$

14.6. Comparison test

$\sum a_n \quad a_n \geq 0$ for all n.

(i) If $\sum a_n$ converges & $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges

(ii) If $\sum a_n = +\infty$ & $b_n \geq a_n$ for all n then $\sum b_n = \infty$

Pf. (i) for $n \geq m$ we have

$$\left|\sum_{k=m}^{n} b_k\right| \le \sum_{k=m}^{n} |b_k| \le \sum_{k=m}^{n} a_k$$

$\uparrow$

Δ ineq

Since Σa_n converges, it satisfies Cauchy

$\Rightarrow$ Σb_n satisfies Cauchy

Hence converges.

(ii) Let $s_n = \sum_{k=1}^{n} a_k$

$t_n = \sum_{k=1}^{n} b_k$

$b_k \ge a_k \Rightarrow t_n \ge s_n$

But $\lim s_n = \infty$

$\Rightarrow \lim t_n = \infty$

Def Σa_n is said to converge absolutely if $\Sigma |a_n|$ converges

Cor Absolutely convgnt series are convgnt

pf suppose Σb_n converges absolutely

Let $a_n = |b_n|$, so Σa_n converges.

But $|b_n| \le a_n$, so Σb_n converges

Ratio test. (not as powerful as root test)

$\sum a_n \qquad a_n \neq 0.$

(i) conv absolutely if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$

(ii) diverges if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$

(iii) Otherwise no info.

$$\liminf |\div| \leq 1 \leq \limsup |\div|$$

Pf: in 10 mins.

Root test.

$\sum a_n$. $\quad \alpha = \limsup |a_n|^{\frac{1}{n}}$

Then $\sum a_n$

(i) converges absolutely if $\alpha < 1$

(ii) diverges if $\alpha > 1$.

(iii) $\alpha = 1$. no info

Pf. (i)

Suppose $\alpha < 1$. pick ε s.t. $\alpha + \varepsilon < 1$.

Then $\exists N$ s.t.

$$\alpha - \varepsilon < \sup\{|a_n|^{\frac{1}{n}} : n > N\} < \alpha + \varepsilon$$

$\Rightarrow \quad |a_n|^{\frac{1}{n}} < \alpha + \varepsilon$ for $n > N$

so $\quad |a_n| < (\alpha+\varepsilon)^n$ for $n > N$.

The geometric series $\sum_{n=N+1}^{\infty} (\alpha+\varepsilon)^n$ converges

The comparison test $\Rightarrow$

$\sum_{n=N+1}^{\infty} a_n$ also converges

$\Rightarrow \sum a_n$ converges.

(ii) If $\alpha > 1$, then a subseq of $|a_n|^{\frac{1}{n}}$ has limit $\alpha > 1$

$\Rightarrow |a_n| > 1$ for infinitely many n.

$\Rightarrow a_n$ does not converge to 0

$\Rightarrow \sum a_n$ diverges

(iii) $\sum \frac{1}{n}$ & $\sum \frac{1}{n^2}$

Pf of ratio test

Let $\alpha = \limsup |a_n|^{\frac{1}{n}}$

$$\Rightarrow \liminf \left|\frac{a_{n+1}}{a_n}\right| \le \alpha \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\alpha < 1$

Root test $\Rightarrow \sum a_n$ converges

If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\alpha > 1$

Root test $\Rightarrow \sum a_n$ diverges

Example.

$$\sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^n = \frac{1}{9} - \frac{1}{27} + \frac{1}{81} \cdots$$

Example. $\sum \frac{n}{n^2+3}$

$\lim \frac{a_{n+1}}{a_n} = 1$ so ratio test does not work

But $\liminf \frac{a_{n+1}}{a_n} \le \liminf a_n^{\frac{1}{n}} \le \limsup a_n^{\frac{1}{n}}$
$\le \limsup \frac{a_{n+1}}{a_n}$

$\Rightarrow \lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$

But. $\frac{n}{n^2+3} \geq \frac{n}{n^2+3n^2} \geq \frac{1}{4n}$

But $\sum \frac{1}{n}$ diverges to ∞.

Example. $\sum \frac{1}{n^2+1}$

ratio & root test fails.

$\frac{1}{n^2+1} \le \frac{1}{n^2}$

since $\sum \frac{1}{n^2}$ converges, $\sum \frac{1}{n^2+1}$ converges.

Example. $\sum \frac{n}{3^n}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{3} < 1$. converges.

Example. $\sum \left(\frac{2}{(-1)^n - 3}\right)^n$

$\limsup |a_n|^{\frac{1}{n}} = \limsup \left|\frac{2}{(-1)^n-3}\right| = 1$

But $\lim_{n\to\infty} \frac{2}{(-1)^n-3} \neq 0$.

Example. $\sum_{n=0}^{\infty} 2^{(-1)^n - n}$

$$\left| 2^{(-1)^n - n} \right| \le 2^{-1-n} = \frac{1}{2^{n+1}}$$

so $\sum_{n=0}^{\infty} 2^{(-1)^n} - n$ converges.

Example. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

ratio, root does not work.

comparison does not work.

We need to study alternating series.