Lecture Notes on Alternating Series

Harmonic Series and p-Series

Example 1 (Harmonic Series)**.** *The harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

Proof (Method 1 - Integral Test):

$$\sum_{n=1}^{N} \frac{1}{n} > \int_{1}^{N+1} \frac{1}{x} dx = \log x \Big|_{1}^{N+1} = \log(N+1)$$

As $N \to \infty$, $\log(N+1) \to +\infty$, so $\sum \frac{1}{n} = +\infty$.

Proof (Method 2 - Grouping Terms):

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

$$\geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$$

Theorem 1 (p-Series Convergence)**.** *The series* $\sum \frac{1}{n^p}$*:*

- *Converges if* $p > 1$
- *Diverges if* $0 < p \leq 1$

Proof for $p > 1$**:**

$$\sum_{k=1}^{n} \frac{1}{k^p} \leq 1 + \int_{1}^{n} \frac{1}{x^p} dx = 1 + \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right) \leq 1 + \frac{1}{p-1}$$

So the partial sums are bounded above.

Proof for $0 < p \leq 1$**:** Since $\frac{1}{n^p} \geq \frac{1}{n}$ for $0 < p \leq 1$ and $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^p}$ also diverges by comparison.

Alternating Series

Theorem 2 (Alternating Series Test)**.** *If* $a_n > 0$ *for all* n*,* (a_n) *is decreasing, and* $\lim_{n\to\infty} a_n = 0$*, then the alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Example 2. *The alternating harmonic series* $\sum \frac{(-1)^n}{n}$ *converges.*

Proof of Alternating Series Test: Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$. For $m > n$:

$$|S_m - S_n| = \left| \sum_{k=n+1}^{m} (-1)^k a_k \right| = |a_{n+1} - a_{n+2} + a_{n+3} - \cdots \pm a_m|$$

Consider two cases:

Case 1: The sum has an even number of terms

$$|S_m - S_n| = (a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdots + (a_{m-1} - a_m) \leq a_{n+1}$$

Case 2: The sum has an odd number of terms

$$|S_m - S_n| = a_{n+1} - (a_{n+2} - a_{n+3}) - \cdots - (a_{m-1} - a_m) \leq a_{n+1}$$

In both cases, $|S_m - S_n| \leq a_{n+1}$. Since $\lim_{n\to\infty} a_n = 0$, the sequence (S_n) is Cauchy and therefore converges.