1 Continuity of Functions

Let f be a function whose domain is a subset of $\mathbb{R}$.

Definition 1. *Let $x_0 \in dom(f)$. The function f is said to be **continuous at** x_0 if for every sequence (x_n) in $dom(f)$ that converges to x_0, we have*

$$\lim_{n\to\infty} f(x_n) = f(x_0).$$

*The function f is said to be **continuous** if it is continuous at all $x \in dom(f)$.*

Theorem 1 (ε-δ Criterion)**.** *Let $x_0 \in dom(f)$. Then f is continuous at x_0 if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in dom(f)$ with $|x - x_0| < \delta$, we have*

$$|f(x) - f(x_0)| < \varepsilon.$$

Proof. ($\Leftarrow$) Suppose the ε-δ condition holds. Let (x_n) be a sequence in $\mathrm{dom}(f)$ with $x_n \to x_0$. For any $\varepsilon > 0$, choose $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Since $x_n \to x_0$, there exists N such that for all $n > N$, $|x_n - x_0| < \delta$. Then $|f(x_n) - f(x_0)| < \varepsilon$ for all $n > N$, so $f(x_n) \to f(x_0)$.

($\Rightarrow$) We prove the contrapositive. Suppose the ε-δ condition fails. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$, there is some $x \in \mathrm{dom}(f)$ with $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \varepsilon_0$. Taking $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$, we obtain a sequence (x_n) in $\mathrm{dom}(f)$ with $|x_n - x_0| < \frac{1}{n}$ but $|f(x_n) - f(x_0)| \geq \varepsilon_0$. Then $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$, so f is not continuous at x_0. $\square$

2 Examples

Example 1. *Let $f(x) = 2x^2 + 1$. Then f is continuous on $\mathbb{R}$.*

Proof. Let $x_0 \in \mathbb{R}$. Then

$$f(x) - f(x_0) = 2x^2 - 2x_0^2 = 2(x + x_0)(x - x_0).$$

For any $\varepsilon > 0$, let

$$\delta = \min\left\{1, \frac{\varepsilon}{2(2|x_0| + 1)}\right\}.$$

If $|x - x_0| < \delta$, then

$$\begin{aligned}
|f(x) - f(x_0)| &= 2|x + x_0||x - x_0| \\
&\leq 2(|x| + |x_0|)\delta \\
&\leq 2(2|x_0| + \delta)\delta \quad (\text{since } |x| \leq |x_0| + \delta < |x_0| + 1) \\
&\leq 2(2|x_0| + 1) \cdot \frac{\varepsilon}{2(2|x_0| + 1)} = \varepsilon.
\end{aligned}$$

Hence f is continuous at x_0. $\square$

3 Properties of Continuous Functions

Theorem 2. *If f is continuous, then:*

1. *$|f|$ is continuous.*
2. *kf is continuous for any $k \in \mathbb{R}$.*

Proof. Let $x_0 \in \mathrm{dom}(f)$.

1. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Then

$$||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)| < \varepsilon.$$

Hence $|f|$ is continuous at x_0.

2. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0)| < \frac{\varepsilon}{|k| + 1}.$$

Then

$$|kf(x) - kf(x_0)| = |k||f(x) - f(x_0)| < |k| \cdot \frac{\varepsilon}{|k| + 1} < \varepsilon.$$

Hence kf is continuous at x_0.

□

Theorem 3. *Let f and g be continuous at x_0. Then:*

1. *$f + g$ is continuous at x_0.*
2. *$f - g$ is continuous at x_0.*
3. *f/g is continuous at x_0 if $g(x_0) \neq 0$.*

Proof. Let (x_n) be a sequence in $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ with $x_n \to x_0$.

1.

$$\begin{aligned}
\lim_{n\to\infty} (f + g)(x_n) &= \lim_{n\to\infty} [f(x_n) + g(x_n)] \\
&= \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) \\
&= f(x_0) + g(x_0) = (f + g)(x_0).
\end{aligned}$$

2. Similar to (i).
3. Standard result from calculus.

□

Theorem 4 (Composition)**.** *If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*