1 Boundedness and Extreme Values of Continuous Functions

Definition 1. *A function f is **bounded** if there exists $M > 0$ such that $|f(x)| \leq M$ for all x in its domain.*

Theorem 1 (Boundedness Theorem)**.** *If f is continuous on $[a, b]$, then f is bounded.*

Proof. Suppose for contradiction that f is not bounded. Then there exists a sequence (x_k) in $[a, b]$ such that $f(x_k) \to \infty$. Since $[a, b]$ is compact, by Bolzano-Weierstrass, there exists a convergent subsequence $(x_{k_l}) \to x_0 \in [a, b]$. By continuity:

$$\lim_{l\to\infty} f(x_{k_l}) = f(x_0) < \infty,$$

which contradicts $f(x_k) \to \infty$. □

Theorem 2 (Extreme Value Theorem)**.** *If f is continuous on $[a, b]$, then f attains its maximum and minimum values on $[a, b]$.*

Proof. Let $M = \sup\{f(x) : x \in [a, b]\}$, which is finite by the boundedness theorem. There exists a sequence (y_n) in $[a, b]$ such that:

$$M - \frac{1}{n} \leq f(y_n) \leq M.$$

Then $\lim_{n\to\infty} f(y_n) = M$. By Bolzano-Weierstrass, there exists a convergent subsequence $(y_{n_k}) \to y_0 \in [a, b]$. By continuity:

$$\lim_{k\to\infty} f(y_{n_k}) = f(y_0) = M.$$

Thus f attains its maximum at y_0. The proof for the minimum is similar. □

Theorem 3 (Intermediate Value Theorem)**.** *If f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then for any y between $f(a)$ and $f(b)$, there exists $x \in (a, b)$ such that $f(x) = y$.*

Proof. Assume without loss of generality that $f(a) < f(b)$. Let y satisfy $f(a) < y < f(b)$. Define:

$$S = \{x \in [a, b] : f(x) < y\}.$$

Since $a \in S$, S is non-empty. Let $x_0 = \sup S \leq b$.

Since $x_0 - \frac{1}{n}$ is not an upper bound for S, there exists $x_n \in S$ such that:

$$x_0 - \frac{1}{n} \leq x_n \leq x_0.$$

Then $x_n \to x_0$, and by continuity $f(x_n) \to f(x_0)$. Since $f(x_n) < y$ for all n, we have $f(x_0) \leq y$.

Also, $x_0 + \frac{1}{n} > x_0$, so $x_0 + \frac{1}{n} \notin S$, hence $f(x_0 + \frac{1}{n}) \geq y$. Taking the limit as $n \to \infty$, we get $f(x_0) \geq y$.

Therefore, $f(x_0) = y$. □

Corollary 3.1. *If f is continuous and I is an interval, then $f(I)$ is either a point or an interval.*

Example. *Let $f : [0, 1] \to [0, 1]$ be continuous. Then there exists $x_0 \in [0, 1]$ such that $f(x_0) = x_0$ (fixed point).*

Proof. Define $g(x) = f(x) - x$. Then:

$$g(0) = f(0) \geq 0, \quad g(1) = f(1) - 1 \leq 0.$$

If $g(0) = 0$, then $f(0) = 0$ and $x_0 = 0$. If $g(1) = 0$, then $f(1) = 1$ and $x_0 = 1$. Otherwise, $g(1) < 0 < g(0)$, so by the Intermediate Value Theorem, there exists $x_0 \in (0, 1)$ such that $g(x_0) = 0$, i.e., $f(x_0) = x_0$. □