1 Operations on Continuous Functions

Theorem 1 (Composition)**.** *If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Proof. Let (x_n) be a sequence with $x_n \to x_0$. By continuity of f:

$$f(x_n) \to f(x_0).$$

By continuity of g:

$$g(f(x_n)) \to g(f(x_0)).$$

Thus $(g \circ f)(x_n) \to (g \circ f)(x_0)$, so $g \circ f$ is continuous at x_0. □

Theorem 2. *If f and g are continuous at x_0, then* $\max(f, g)$ *is continuous at x_0.*

Proof. We have the identity:

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|.$$

Since f and g are continuous at x_0, so are $f + g$ and $f - g$. The absolute value function is continuous, so $|f - g|$ is continuous. Therefore, $\max(f, g)$ is continuous at x_0. □

2 Inverse Functions

Theorem 3. *Let f be a continuous strictly increasing function on some interval I. Then $f(I) = J$ is an interval. Let $f^{-1} : J \to I$ be the inverse of f.*

Proof. Since f is strictly increasing and continuous, it is one-to-one, and by the Intermediate Value Theorem, its image $J = f(I)$ is an interval.

For any $y \in J$, there exists a unique $x \in I$ such that $f(x) = y$. Define $f^{-1}(y) = x$.

To show f^{-1} is strictly increasing: let $y_1, y_2 \in J$ with $y_1 > y_2$, and let $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$. If $x_1 \leq x_2$, then since f is strictly increasing, $f(x_1) \leq f(x_2)$, i.e., $y_1 \leq y_2$, which is a contradiction. Hence $x_1 > x_2$, so f^{-1} is strictly increasing.

The continuity of f^{-1} follows from the fact that the inverse of a continuous strictly monotone function on an interval is continuous. □

3 Characterization of One-to-One Continuous Functions

Theorem 4. *Let f be a one-to-one continuous function on an interval I. Then f is strictly monotonic (either strictly increasing or strictly decreasing).*

Proof. Here one-to-one means: $f(x) = f(x') \Rightarrow x = x'$.

Suppose f is not strictly monotonic. Then there exist $a < b < c$ in I such that either:

1. $f(b) < \min\{f(a), f(c)\}$, or
2. $f(b) > \max\{f(a), f(c)\}$.

In either case, by the Intermediate Value Theorem, f takes some value twice, contradicting injectivity. Hence f must be strictly monotonic. $\square$

4 Uniform Continuity

Definition 1. *Let $S \subseteq dom(f)$. The function f is **continuous on** S if for every $x_0 \in S$ and every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in S$ with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \epsilon$.*

Here, δ may depend on both ϵ and x_0.

Definition 2. *A function f is **uniformly continuous on** S if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, x' \in S$ with $|x - x'| < \delta$, we have $|f(x) - f(x')| < \epsilon$.*

Here, δ depends only on ϵ, not on the particular points in S.

Example. *The function $f(x) = \frac{1}{x}$ on $(0, 1)$ is continuous but not uniformly continuous.*

Theorem 5. *If f is continuous on $[a, b]$ (where $a, b \in \mathbb{R}$), then f is uniformly continuous on $[a, b]$.*

Proof. Suppose for contradiction that f is not uniformly continuous on $[a, b]$. Then there exists $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$, there exist $x_n, y_n \in [a, b]$ with:

$$|x_n - y_n| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

By Bolzano-Weierstrass, there exists a convergent subsequence $(x_{n_k}) \to x_0 \in [a, b]$. Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$, we also have $y_{n_k} \to x_0$.

By continuity:

$$\lim_{k \to \infty} f(x_{n_k}) = f(x_0) = \lim_{k \to \infty} f(y_{n_k}).$$

But this contradicts $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0$ for all k. $\square$