1 Uniform Continuity on a Closed Interval

Theorem 1. *If f is continuous on $[a, b]$, where $a, b \in \mathbb{R}$, then f is uniformly continuous on $[a, b]$.*

Proof. Suppose not. Then

$$\exists \epsilon > 0, \forall \delta > 0, \exists x, y \in [a, b] \text{ such that } |x - y| < \delta \text{ but } |f(x) - f(y)| \geq \epsilon.$$

Let $\delta = \frac{1}{n}$. Then there exist sequences $(x_n), (y_n) \subset [a, b]$ such that

$$|x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \epsilon.$$

By the Bolzano–Weierstrass theorem, there exists a convergent subsequence $(x_{n_k}) \to x_\infty \in [a, b]$. Then

$$\lim_{k \to \infty} y_{n_k} = x_\infty.$$

By continuity,

$$f(x_\infty) = \lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(y_{n_k}),$$

so

$$\lim_{k \to \infty} (f(x_{n_k}) - f(y_{n_k})) = 0,$$

contradicting $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$. □

2 Connectedness of Intervals

Lemma 1. *Let $A \subset [a, b]$ be both closed and open. Then $A = \emptyset$ or $A = [a, b]$.*

Proof. Suppose $A \neq \emptyset$ and $A \neq [a, b]$. Let $s = \sup A$. Since A is closed, $s \in A$. But A is open, so there exists an open interval around s contained in A, contradicting that $s = \sup A$. □

3 Compactness of Closed Intervals

Theorem 2. *The interval $[a, b]$ is compact.*

Proof. Let $\{U_i\}$ be an open cover of $[a, b]$. Define

$$S = \{s \in [a, b] : [a, s] \text{ can be covered by finitely many } U_i\}.$$

Then $a \in S$, so $S \neq \emptyset$. Let $M = \sup S$. Then $M \in [a, b]$, and there exists U_{i_0} such that $M \in U_{i_0}$. Pick $\epsilon > 0$ such that $(M - \epsilon, M + \epsilon) \subset U_{i_0}$. Then $\exists s \in S$ with $s > M - \epsilon$, so $[a, s]$ is finitely covered, and adding U_{i_0} covers $[a, M + \frac{\epsilon}{2}]$, contradicting $M = \sup S$. Hence $b \in S$. □

4 Alternative Proof of Uniform Continuity

Suppose f is continuous at $x_0 \in [a,b]$. Then

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } x \in [a,b],\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{\epsilon}{2}.$$

Now, for any $x_0' \in [a,b]$, define $\delta' = \frac{1}{2}\delta(x_0)$. Then if $|x - x_0'| < \delta'$, we have

$$|f(x) - f(x_0')| \leq |f(x) - f(x_0)| + |f(x_0) - f(x_0')|.$$

But $|x_0 - x_0'| < \frac{\delta}{2} = \delta'$, so $|f(x_0) - f(x_0')| < \frac{\epsilon}{2}$, and

$$|x - x_0| \leq |x - x_0'| + |x_0' - x_0| \leq \delta' + \frac{\delta}{2} = \delta,$$

so $|f(x) - f(x_0)| < \frac{\epsilon}{2}$, hence $|f(x) - f(x_0')| < \epsilon$.

Thus, for each $x_0 \in [a,b]$, the interval $(x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}) \cap [a,b]$ is an open cover of $[a,b]$. By compactness, there exists a finite subcover $U_1, \ldots, U_N$. For each U_i, let δ_i be the corresponding δ, and take $\delta = \min\{\delta_1, \ldots, \delta_N\}$. Then f is uniformly continuous.

Example 1. *The function $f(x) = \frac{1}{x}$ is uniformly continuous on $[\frac{1}{2}, 5]$.*

5 Non-Uniform Continuity

Example 2. *The function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, \infty)$.*

Proof. Suppose it is. Let $\epsilon = 1$. Then $\exists \delta > 0$ such that $|x - x'| < \delta \Rightarrow |\frac{1}{x} - \frac{1}{x'}| < 1$. Choose N such that $\frac{2}{N} < \delta$. Let $x = \frac{1}{N}, x' = \frac{1}{N+2}$. Then $|x - x'| < \delta$, but $|\frac{1}{x} - \frac{1}{x'}| = 2 > 1$, a contradiction. □

Theorem 3. *If f is uniformly continuous on S and $(s_n) \subset S$ is Cauchy, then $f(s_n)$ is Cauchy.*

Proof. Let $\epsilon > 0$. Choose $\delta > 0$ such that $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon$. Since (s_n) is Cauchy, $\exists N$ such that $n, m > N \Rightarrow |s_n - s_m| < \delta$, so $|f(s_n) - f(s_m)| < \epsilon$. □

Example 3. *For $f(x) = \frac{1}{x}$ on $(0, \infty)$, the sequence $s_n = \frac{1}{n}$ is Cauchy, but $f(s_n) = n$ is not Cauchy. Hence f is not uniformly continuous.*

6 Extension of Uniformly Continuous Functions

Theorem 4. *If f on (a,b) is uniformly continuous, then it can be extended to a continuous function $\bar{f}$ on $[a,b]$.*

Proof. Let $(s_n) \subset (a,b)$ be a sequence converging to a. Since f is uniformly continuous, $f(s_n)$ is Cauchy and hence converges to some $y \in \mathbb{R}$. Define $\bar{f}(a) = y$. Similarly for b. This definition is independent of the sequence chosen. One can then verify that $\bar{f}$ is continuous on $[a,b]$. □