23. Power Series

Consider a power series of the form

$$\sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{R},$$

where x is the variable and a_n are the coefficients.

Note: The series always converges when $x = 0$ (with the convention $0^0 = 1$).

Theorem 1 (Radius of Convergence)**.** *Let*

$$\beta = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \quad \textit{and} \quad R = \frac{1}{\beta}.$$

Then the power series $\sum a_n x^n$:

- *Converges absolutely for $|x| < R$*
- *Diverges for $|x| > R$*

*The number R is called the **radius of convergence**.*

Proof. Apply the root test:

$$\limsup_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = \limsup_{n\to\infty} |x| \cdot |a_n|^{\frac{1}{n}} = |x| \cdot \beta.$$

Consider three cases:
Case 1: $0 < \beta < \infty \Leftrightarrow 0 < R < +\infty$
If $|x| < R$, then

$$|x| \cdot \beta < R \cdot \frac{1}{R} = 1 \Rightarrow \sum a_n x^n \text{ converges absolutely.}$$

If $|x| > R$, then

$$|x| \cdot \beta > 1 \Rightarrow \sum a_n x^n \text{ diverges.}$$

Case 2: $\beta = 0 \Rightarrow R = +\infty$
Then $|x| \cdot \beta = 0 < 1$ for all x, so the series converges for all $x \in \mathbb{R}$.
Case 3: $\beta = +\infty \Rightarrow R = 0$
For any $x \neq 0$, $|x| \cdot \beta = +\infty > 1$, so the series diverges for all $x \neq 0$. $\square$

Remark 1. *The ratio test can also be applied to determine the radius of convergence, but we skip the details here.*

Example 1.

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Using the ratio test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{n+1} \to 0 \quad \textit{as } n \to \infty,$$

so $R = +\infty$. In fact, this series represents e^x.

Example 2.

$$\sum_{n=0}^{\infty} x^n$$

Here $R = 1$. When $x = \pm 1$, the series diverges.

Example 3.

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

Radius of convergence $R = 1$. At the endpoints:

- $x = 1$: $\sum \frac{1}{n}$ *diverges (harmonic series)*
- $x = -1$: $\sum \frac{(-1)^n}{n}$ *converges (alternating harmonic series)*

Example 4.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$$

Radius of convergence $R = 1$. The series converges when $x = \pm 1$.

Example 5.

$$\sum_{n=0}^{\infty} n! x^n$$

Here $R = 0$, and the series diverges for all $x \neq 0$.

Example 6.

$$\sum_{n=0}^{\infty} 2^{-n} x^{3^n}$$

To find the radius of convergence:

$$\limsup_{n\to\infty} \left(2^{-n}\right)^{\frac{1}{3^n}} = 2^{-\limsup \frac{n}{3^n}} = 2^0 = 1,$$

so $R = 1$.

Definition 1 (General Power Series)**.** *More generally, we can consider power series centered at* x_0*:*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The radius of convergence R *is defined similarly, and the series converges absolutely for* $|x - x_0| < R$*, diverges for* $|x - x_0| > R$*.*