Convergence of Functions

Definition 1 (Pointwise Convergence)**.** *Let* (f_n) *be a sequence of functions defined on* $S \subseteq \mathbb{R}$*. We say* (f_n) *converges* ***pointwise*** *to* f *if*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \textit{for all } x \in S.$$

Example 1. *Let* $f_n(x) = (1 - |x|)^n$ *for* $x \in (-1, 1)$*. Then*

$$\lim_{n\to\infty} f_n(x) = 0 \quad \textit{for all } |x| < 1.$$

So on $(-1, 1)$*,* $f_n \to f$ *pointwise, where* $f(x) = 0$*.*

Remark 1. *Pointwise convergence means:*

$$\forall x \in S, \forall \epsilon > 0, \exists N \textit{ such that } n > N \Rightarrow |f_n(x) - f(x)| < \epsilon.$$

Note: N *may depend on* x*.*

Definition 2 (Uniform Convergence)**.** (f_n) *converges to* f ***uniformly*** *on* S *if*

$$\forall \epsilon > 0, \exists N \textit{ such that } n > N \Rightarrow |f_n(x) - f(x)| < \epsilon \quad \textit{for all } x \in S.$$

Remark 2. *For uniform convergence,* N *depends only on* ϵ *and not on* x*.*

Example 2. *Let* $f_n(x) = (1 - |x|)^n$ *for* $x \in (-1, 1)$*, and define*

$$f(x) = \begin{cases} 0 & \textit{if } x \neq 0 \\ 1 & \textit{if } x = 0 \end{cases}$$

Then $f_n \to f$ *pointwise but not uniformly.*

Proof. Suppose for contradiction that $f_n \to f$ uniformly. Then for $\epsilon = \frac{1}{2}$, there exists N such that for all $n > N$ and all $x \in (-1, 1)$,

$$|f_n(x) - f(x)| < \frac{1}{2}.$$

In particular, for $x \neq 0$, $|(1 - |x|)^n| < \frac{1}{2}$ for all $n > N$.

Take $x = 1 - 2^{-\frac{1}{N+2}} \in (-1, 1)$. Then

$$|(1 - |x|)^n| = 2^{-\frac{n}{N+2}}.$$

For $n = N + 1$, we have $\frac{n}{N+2} < 1$, so

$$2^{-\frac{n}{N+2}} > 2^{-1} = \frac{1}{2},$$

which is a contradiction. □

Example 3. *Let $f_n(x) = \frac{1}{n}\sin(nx)$. Then $f_n \to 0$ uniformly for all $x \in \mathbb{R}$.*

Example 4. *Let $f_n(x) = nx^n$ for $x \in [0, 1)$. Then $f_n \to 0$ pointwise but not uniformly.*

Take $x_n = n^{-\frac{1}{n}}$. Then

$$f_n(x_n) = n \cdot (n^{-\frac{1}{n}})^n = n \cdot n^{-1} = 1 \not\to 0.$$

Example 5. *Let $f_n(x) = x^n$ for $x \in [0, 1]$, and define*

$$f(x) = \begin{cases} 0 & \textit{if } x \neq 1 \\ 1 & \textit{if } x = 1 \end{cases}$$

Then $f_n \to f$ pointwise but not uniformly.

Theorem 1 (Uniform Limit of Continuous Functions)**.** *If (f_n) is a sequence of continuous functions that converges uniformly to f on S, then f is continuous on S.*

Proof. Let $x_0 \in S$. For any $x \in S$ and $\epsilon > 0$, we have:

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

Since $f_n \to f$ uniformly, there exists N such that for all $n > N$ and all $x \in S$,

$$|f(x) - f_n(x)| < \frac{\epsilon}{3}.$$

Since f_n is continuous at x_0, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}.$$

Therefore, for $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

So f is continuous at x_0. □

Remark 3 (Reformulation of Uniform Convergence)**.** *$(f_n) \to f$ uniformly on $S \subseteq \mathbb{R}$ if and only if*

$$\lim_{n\to\infty} \sup_{x\in S} |f(x) - f_n(x)| = 0.$$

Example 6. *Let $f_n(x) = \frac{x}{1+nx^2}$ and $f(x) = 0$. Then $f_n \to f$ uniformly on $\mathbb{R}$.*

To find the supremum, consider the derivative:

$$\left(\frac{x}{1+nx^2}\right)' = \frac{1-nx^2}{(1+nx^2)^2} = 0 \Rightarrow x = \pm\frac{1}{\sqrt{n}}.$$

Then

$$\sup_{x\in\mathbb{R}} |f_n(x)| = \frac{1/\sqrt{n}}{1+n(1/n)} = \frac{1}{2\sqrt{n}} \to 0.$$

So $\sup_{x\in S} |f_n(x)| \to 0$*, and the convergence is uniform.*