More on Uniform Convergence

We will use the following facts about integration:

1. If g, h are integrable on $[a, b]$ and $g \leq h$, then
$$\int_a^b g(x)\,dx \leq \int_a^b h(x)\,dx.$$
2. If g is integrable on $[a, b]$, then
$$\left|\int_a^b g(x)\,dx\right| \leq \int_a^b |g(x)|\,dx.$$

Theorem 1 (Interchange of Limit and Integral)**.** *Let f_n be continuous on $[a, b]$ and suppose $f_n \to f$ uniformly. Then*
$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. By the previous lecture, f is continuous (hence integrable) and $f_n - f$ is continuous and integrable.

For any $\epsilon > 0$, there exists N such that for all $n > N$ and all $x \in [a, b]$,
$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}.$$
Then
$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| = \left|\int_a^b (f_n(x) - f(x))\,dx\right| \leq \int_a^b |f_n(x) - f(x)|\,dx \leq \int_a^b \frac{\epsilon}{b-a}\,dx = \epsilon.$$
□

Definition 1 (Uniformly Cauchy Sequence)**.** *A sequence (f_n) of functions on $S \subseteq \mathbb{R}$ is **uniformly Cauchy** on S if*

$\forall \varepsilon > 0, \exists N$ *such that* $|f_n(x) - f_m(x)| < \varepsilon$ *for all* $m, n > N$ *and all* $x \in S$.

Theorem 2 (Completeness for Uniform Convergence)**.** *Let (f_n) be a sequence of functions on S. If (f_n) is uniformly Cauchy, then there exists f on S such that $f_n \to f$ uniformly.*

Proof. First, define f. For each $x_0 \in S$, the sequence $(f_n(x_0))$ is Cauchy in $\mathbb{R}$, hence converges. Define

$$f(x_0) = \lim_{n \to \infty} f_n(x_0).$$

Now show uniform convergence. Given $\varepsilon > 0$, there exists N such that for all $m, n > N$ and all $x \in S$,

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}.$$

Fix $x \in S$ and $m > N$. Taking $n \to \infty$ and using continuity of the absolute value function:

$$\lim_{n \to \infty} |f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2} \Rightarrow |f(x) - f_m(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus $f_m \to f$ uniformly. $\square$

Example 1 (Weierstrass Function)**.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, piecewise linear function with period 4, defined by:*

$$g(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 2 - x & \text{for } 1 \leq x \leq 3 \\ x - 4 & \text{for } 3 \leq x \leq 4 \end{cases}$$

Define $g_n(x) = g(4^n x)$ and consider the series

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n g_n(x).$$

Let $f_n(x) = \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k g_k(x)$. Then for $n > m$,

$$|f_n(x) - f_m(x)| = \left| \sum_{k=m+1}^{n} \left(\frac{3}{4}\right)^k g_k(x) \right| \leq \sum_{k=m+1}^{n} \left(\frac{3}{4}\right)^k.$$

Since $\sum \left(\frac{3}{4}\right)^k$ converges, (f_n) is uniformly Cauchy. Thus $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n g_n(x)$ converges uniformly to a continuous function that is nowhere differentiable.

Theorem 3 (Continuity of Uniform Limits of Series)**.** *Let $\sum_{k=0}^{\infty} g_k$ be a series of functions defined on $S \subseteq \mathbb{R}$. If each g_k is continuous on S and the series converges uniformly on S, then the sum represents a continuous function on S.*

Proof. Let $f_n = \sum_{k=1}^{n} g_k$. Then each f_n is continuous and $f_n \to f$ uniformly, so f is continuous. $\square$

Theorem 4 (Weierstrass M-Test)**.** *Let (M_k) be a sequence of non-negative real numbers with $\sum M_k < \infty$. If $|g_k(x)| \le M_k$ for all $x \in S$, then $\sum g_k$ converges uniformly on S.*

Proof. Check the Cauchy criterion. Since $\sum M_k$ converges, for any $\epsilon > 0$, there exists N such that for all $n \ge m > N$,

$$\sum_{k=m}^{n} M_k < \epsilon.$$

Then for all $x \in S$,

$$\left| \sum_{k=m}^{n} g_k(x) \right| \le \sum_{k=m}^{n} |g_k(x)| \le \sum_{k=m}^{n} M_k < \epsilon.$$

So the series converges uniformly. $\square$

Example 2. *Consider $\sum_{n=1}^{\infty} 2^{-n} x^n$ on $(-2, 2)$. The radius of convergence is $R = 2$.*

For any $0 < a < 2$, on $[-a, a]$ we have:

$$|2^{-n} x^n| \le 2^{-n} a^n = \left(\frac{a}{2}\right)^n.$$

Since $\sum \left(\frac{a}{2}\right)^n$ converges, by the Weierstrass M-test, the series converges uniformly on $[-a, a]$ to a continuous function.

However, the convergence is not uniform on $(-2, 2)$ because:

$$\sup \left\{ |2^{-n} x^n| : x \in (-2, 2) \right\} = 1 \not\to 0.$$

Remark 1. *If $\sum g_n$ converges uniformly on S, then $\lim_{n\to\infty} \sup\{|g_n(x)| : x \in S\} = 0$.*

Proof. Since $\sum g_n$ converges uniformly, it satisfies the Cauchy criterion. For any $\epsilon > 0$, there exists N such that for all $n > m > N$ and all $x \in S$,

$$\left| \sum_{k=m}^{n} g_k(x) \right| < \epsilon.$$

In particular, for $n > N$, taking $m = n$ gives $|g_n(x)| < \epsilon$ for all $x \in S$, so $\sup\{|g_n(x)| : x \in S\} < \epsilon$. $\square$

Example 3 (Counterexample to Converse of M-Test)**.** *There exist uniformly convergent series for which no convergent majorant series exists.*

Take $S = \mathbb{R}$*,* $g_1(x) = x$*, and* $g_n(x) = 0$ *for* $n \neq 1$*. This series converges uniformly but no sequence* M_n *with* $\sum M_n < \infty$ *can majorize it.*

Even for compact S*, consider* $g_n(x) = \frac{1}{n}\sin(nx)$*. Then* $\sum g_n$ *converges uniformly (by Dirichlet's test), but* $\sum \frac{1}{n} = \infty$*.*

Theorem 5 (Dirichlet's Test for Uniform Convergence)**.** *Let* $(a_n(x))$ *and* $(b_n(x))$ *be sequences of functions on* S *such that:*

1. *The partial sums* $A_N(x) = \sum_{n=1}^{N} a_n(x)$ *are uniformly bounded on* S*.*
2. $b_n(x) \to 0$ *uniformly on* S*.*
3. $(b_n(x))$ *is monotone in* n *for each fixed* x*.*

Then $\sum a_n(x)b_n(x)$ *converges uniformly on* S*.*

Remark 2. *For trigonometric series, we have the identity:*

$$\sum_{n=1}^{N} \sin(nx) = \frac{\sin\left(\frac{Nx}{2}\right)\sin\left(\frac{(N+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)},$$

which shows that the partial sums of $\sum \sin(nx)$ *are uniformly bounded away from multiples of* 2π*.*