Differentiation and Integration of Power Series

Theorem 1 (Uniform Convergence on Compact Intervals)**.** *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ (or $R = +\infty$). If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.*

Proof. For $|x| \leq R_1$, we have:

$$|a_n x^n| \leq |a_n| R_1^n.$$

But $\sum |a_n| x^n$ has the same radius of convergence as $\sum a_n x^n$, so $\sum |a_n| R_1^n$ converges. By the Weierstrass M-test, $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$. Hence the limit is continuous. $\square$

Corollary 1. *$\sum a_n x^n$ converges to a continuous function on $(-R, R)$, where $R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$.*

Lemma 1 (Radius of Convergence of Derived Series)**.** *If $\sum a_n x^n$ has radius of convergence R, then both*

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \quad \textit{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R.

Proof. Let $\beta = \limsup |a_n|^{\frac{1}{n}}$, so $R = \frac{1}{\beta}$. Then:

$$\limsup (n|a_n|)^{\frac{1}{n}} = \lim_{n\to\infty} n^{\frac{1}{n}} \cdot \limsup |a_n|^{\frac{1}{n}} = 1 \cdot \beta = \beta.$$

Similarly,

$$\limsup \left(\frac{|a_n|}{n+1} \right)^{\frac{1}{n}} = \lim_{n\to\infty} (n+1)^{-\frac{1}{n}} \cdot \beta = 1 \cdot \beta = \beta.$$

Thus both derived series have radius of convergence R. $\square$

Theorem 2 (Term-by-Term Integration)**.** *Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then for $|x| < R$,*

$$\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Proof. Suppose $x < 0$ (the case $x > 0$ is similar). On the interval $[x, 0]$, the series $\sum a_n t^n$ converges uniformly to $f(t)$. Therefore:

$$\int_x^0 f(t)\,dt = \lim_{n\to\infty} \int_x^0 \sum_{k=0}^n a_k t^k\,dt = \lim_{n\to\infty} \sum_{k=0}^n a_k \int_x^0 t^k\,dt = -\lim_{n\to\infty} \sum_{k=0}^n a_k \frac{1}{k+1} x^{k+1} = -\sum_{k=0}^\infty a_k \frac{1}{k+1} x$$

Rearranging gives the desired result. □

Theorem 3 (Term-by-Term Differentiation)**.** *Let $f(x) = \sum_{n=0}^\infty a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and*

$$f'(x) = \sum_{n=1}^\infty n a_n x^{n-1} \quad \textit{for } |x| < R.$$

Proof. Consider $g(x) = \sum_{n=1}^\infty n a_n x^{n-1}$. This series converges for $|x| < R$ by the lemma. We can integrate g term by term:

$$\int_0^x g(t)\,dt = \sum_{n=1}^\infty a_n x^n = f(x) - a_0 \quad \text{for } |x| < R.$$

Thus, for any $0 < R_1 < R$ and $|x| < R$,

$$f(x) = \int_{-R_1}^x g(t)\,dt + k, \quad \text{where } k = a_0 - \int_{-R_1}^0 g(t)\,dt.$$

Since $g(t)$ is continuous (by uniform convergence on compact sets), by the Fundamental Theorem of Calculus,

$$f'(x) = g(x) = \sum_{n=1}^\infty n a_n x^{n-1} \quad \text{for } |x| < R.$$

□

Example 1 (Geometric Series)**.** *We have:*

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \quad \textit{for } |x| < 1.$$

Differentiating term by term:

$$\sum_{n=1}^\infty n x^{n-1} = \frac{1}{(1-x)^2}.$$

Integrating term by term:

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \int_0^x \frac{1}{1-t}\,dt = -\log(1-x) \quad \textit{for } |x| < 1.$$

Or equivalently:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \textit{for } |x| \leq 1, x \neq -1.$$

It turns out this is also true for $x = 1$, giving:

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

This requires Abel's theorem.

Theorem 4 (Abel's Theorem)**.** *Suppose $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1 and converges at $x = 1$. Then the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on $[0, 1]$.*

Proof. Assume without loss of generality that $f(1) = \sum_{n=0}^{\infty} a_n = 0$ (otherwise consider $f(x) - f(1)$). Let

$$f_n(x) = \sum_{k=0}^{n-1} a_k x^k \quad \text{and} \quad S_n = \sum_{k=0}^{n-1} a_k = f_n(1).$$

Since $f_n \to f$ pointwise on $[0, 1]$ and each f_n is continuous, it suffices to show $f_n \to f$ uniformly on $[0, 1]$.

For $n > m$, we have:

$$\begin{aligned} f_n(x) - f_m(x) &= \sum_{k=m+1}^{n} a_k x^k = \sum_{k=m+1}^{n} (S_k - S_{k-1}) x^k \\ &= \sum_{k=m+1}^{n} S_k x^k - x \sum_{k=m+1}^{n} S_{k-1} x^{k-1} \\ &= \sum_{k=m+1}^{n} S_k x^k - x \sum_{k=m}^{n-1} S_k x^k. \end{aligned}$$

Rearranging gives:

$$f_n(x) - f_m(x) = S_n x^n - S_m x^{m+1} + (1-x) \sum_{k=m+1}^{n-1} S_k x^k.$$

Since $\lim S_n = f(1) = 0$, for any $\epsilon > 0$, there exists N such that for all $n > N$, $|S_n| < \frac{\epsilon}{3}$.

Then for $n > m > N$ and $x \in [0,1]$:

$$|(1-x) \sum_{k=m+1}^{n-1} S_k x^k| \leq \frac{\epsilon}{3}(1-x) \sum_{k=m+1}^{n-1} x^k = \frac{\epsilon}{3}(1-x)x^{m+1} \frac{1-x^{n-m-1}}{1-x} < \frac{\epsilon}{3}.$$

Also, $|S_n x^n| < \frac{\epsilon}{3}$ and $|S_m x^{m+1}| < \frac{\epsilon}{3}$. Therefore:

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for all } x \in [0,1],$$

so (f_n) is uniformly Cauchy on $[0,1]$. □

Remark 1. *We will return to power series later when we discuss Taylor series.*