Lecture Notes: Differentiation

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1 Introduction to Derivatives

Definition 1 (Differentiability)**.** *Let f be a function defined on an open interval containing a. We say that f is **differentiable at** a if the limit*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. In this case, we write

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

*and call $f'(a)$ the **derivative of** f **at** a.*

2 Examples of Derivatives

2.1 Power Function $f(x) = x^2$

Let $f(x) = x^2$. Then for any z:

$$\begin{aligned} f'(z) &= \lim_{x \to z} \frac{x^2 - z^2}{x - z} \\ &= \lim_{x \to z} \frac{(x - z)(x + z)}{x - z} \\ &= \lim_{x \to z} (x + z) = 2z \end{aligned}$$

2.2 General Power Rule

Theorem 1 (Power Rule). *If $f(x) = x^n$ for $n \in \mathbb{R}$, then*

$$f'(x) = nx^{n-1}$$

Proof. For $f(x) = x^n$ and fixed a:

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{x^n - a^n}{x - a} \\
&= \lim_{x \to a} \frac{(x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\
&= \lim_{x \to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\
&= na^{n-1}
\end{aligned}$$

□

3 Properties of Differentiable Functions

Theorem 2 (Differentiability implies Continuity). *If f is differentiable at a, then f is continuous at a.*

Proof.

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left(f(a) + (x - a) \cdot \frac{f(x) - f(a)}{x - a} \right) \\
&= \lim_{x \to a} f(a) + \lim_{x \to a} (x - a) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= f(a) + 0 \cdot f'(a) = f(a)
\end{aligned}$$

□

4 Rules of Differentiation

Theorem 3 (Algebraic Rules). *Let f and g be differentiable at a, and let $c \in \mathbb{R}$. Then:*

1. $(cf)'(a) = cf'(a)$

2. $(f+g)'(a) = f'(a) + g'(a)$

3. $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$

4. $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$, *provided* $g(a) \neq 0$

5 Chain Rule

Theorem 4 (Chain Rule)**.** *If f is differentiable at a and g is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a, and*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Proof. Define the function

$$\ell(h) = \begin{cases} \dfrac{g(f(a)+h) - g(f(a))}{h}, & h \neq 0 \\ g'(f(a)), & h = 0 \end{cases}$$

Then $\ell(h)$ is continuous at 0.

For small h, we have:

$$g(f(a)+h) - g(f(a)) = \ell(h) \cdot h$$

Let $h = f(a+\Delta x) - f(a)$. Since f is continuous at a, h becomes small as $\Delta x \to 0$. Then:

$$g(f(a+\Delta x)) - g(f(a)) = \ell(f(a+\Delta x) - f(a)) \cdot (f(a+\Delta x) - f(a))$$

Therefore:

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{g(f(a+\Delta x)) - g(f(a))}{\Delta x} &= \lim_{\Delta x \to 0} \ell(f(a+\Delta x) - f(a)) \cdot \frac{f(a+\Delta x) - f(a)}{\Delta x} \\ &= \ell(0) \cdot f'(a) \\ &= g'(f(a)) \cdot f'(a) \end{aligned}$$

□