3: Real Numbers

1 Field Structure of Real Numbers

Definition 1.1. A *field* is a set F with two binary operations $+$ and $\cdot$ satisfying:

1. **Associativity:** $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$
2. **Commutativity:** $a + b = b + a$ and $ab = ba$
3. **Identities:** There exist $0, 1 \in F$ with $0 \neq 1$ such that $a + 0 = a$ and $a \cdot 1 = a$
4. **Inverses:** For each $a \in F$, there exists $-a \in F$ such that $a + (-a) = 0$; for each $a \neq 0$, there exists $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$
5. **Distributivity:** $a(b + c) = ab + ac$

Definition 1.2. A *commutative ring* satisfies all field axioms except possibly the existence of multiplicative inverses.

2 Ordered Field Structure

Definition 2.1. An *ordered field* is a field F with a relation $\leq$ satisfying:

1. **Trichotomy:** For any $a, b \in F$, exactly one of $a < b$, $a = b$, or $b < a$ holds
2. **Antisymmetry:** If $a \leq b$ and $b \leq a$, then $a = b$
3. **Transitivity:** If $a \leq b$ and $b \leq c$, then $a \leq c$
4. **Translation invariance:** If $a \leq b$, then $a + c \leq b + c$ for all c
5. **Multiplication by positive elements:** If $a \leq b$ and $0 \leq c$, then $ac \leq bc$

Theorem 2.2 (Basic Field Properties)**.** *In any field, the following hold:*

1. *Cancellation:* $a + c = b + c \Rightarrow a = b$
2. *Zero product:* $a \cdot 0 = 0$ *for all* a
3. *Sign rules:* $(-a)b = -ab$ *and* $(-a)(-b) = ab$
4. *Multiplicative cancellation:* $ac = bc$ *and* $c \neq 0 \Rightarrow a = b$
5. *Zero divisor property:* $ab = 0 \Rightarrow a = 0$ *or* $b = 0$

Proof of (2).

$$\begin{aligned} a \cdot 0 &= a \cdot (0 + 0) \\ &= a \cdot 0 + a \cdot 0 \\ \Rightarrow 0 &= a \cdot 0 \quad \text{(by additive cancellation)} \end{aligned}$$

□

Theorem 2.3 (Ordered Field Properties)**.** *In any ordered field:*

1. *Order reversal:* $a \leq b \Rightarrow -b \leq -a$

2. *Multiplication by negatives:* $a \leq b$ *and* $c \leq 0 \Rightarrow bc \leq ac$

3. *Product of nonnegatives:* $0 \leq a$ *and* $0 \leq b \Rightarrow 0 \leq ab$

4. *Squares are nonnegative:* $0 \leq a^2$

5. *Multiplicative identity:* $0 < 1$

6. *Reciprocal preserves order:* $0 < a \Rightarrow 0 < a^{-1}$

7. *Reciprocal reverses order:* $0 < a < b \Rightarrow 0 < b^{-1} < a^{-1}$

Proof of (4). For any a, by trichotomy either $a \geq 0$ or $a \leq 0$. If $a \geq 0$, then $a^2 \geq 0$ by (3). If $a \leq 0$, then $-a \geq 0$ by (1), so $(-a)^2 \geq 0$, but $(-a)^2 = a^2$. □

3 Absolute Value and Distance

Definition 3.1. The *absolute value* of a number a is defined as:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Definition 3.2. The *distance* between two numbers a and b is defined as:

$$\text{dist}(a, b) = |a - b|$$

Theorem 3.3 (Properties of Absolute Value)**.** *For all numbers* a, b*:*

1. *Nonnegativity:* $|a| \geq 0$

2. *Multiplicativity:* $|ab| = |a| \cdot |b|$

3. *Triangle inequality:* $|a + b| \leq |a| + |b|$

Proof of (3). We have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Adding these inequalities gives:

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

which implies $|a + b| \leq |a| + |b|$. □

Corollary 3.4 (Triangle Inequality for Distance)**.** *For all numbers* a, b, c*:*

$$|a - c| \leq |a - b| + |b - c|$$

That is, $dist(a, c) \leq dist(a, b) + dist(b, c)$.

Proof.

$$|a - c| = |(a - b) + (b - c)| \leq |a - b| + |b - c|$$

by the triangle inequality. □