# Lecture Notes: The Riemann Integral

## 1 Darboux Sums and Integrability

Let $f$ be a bounded function on the interval $[a, b]$.

For a subset $S \subseteq [a, b]$, define

$$M(f, S) = \sup\{f(x) : x \in S\}, \quad m(f, S) = \inf\{f(x) : x \in S\}.$$

**Definition 1.1** (Partition)**.** A **partition** of $[a, b]$ is a finite ordered set

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}.$$

**Definition 1.2** (Darboux Sums)**.** For a partition $P$ as above, the **upper Darboux sum** is

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}),$$

and the **lower Darboux sum** is

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}).$$

It follows immediately that

$$m(f, [a, b])(b - a) \leq L(f, P) \leq U(f, P) \leq M(f, [a, b])(b - a).$$

**Definition 1.3** (Upper and Lower Integrals)**.** The **upper integral** of $f$ is

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\},$$

and the **lower integral** is

$$L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

**Definition 1.4** (Darboux Integrability)**.** We say $f$ is **Darboux integrable** on $[a, b]$ if $L(f) = U(f)$. In that case, the common value is denoted

$$\int_a^b f(x)\, dx = L(f) = U(f).$$

**Example 1.5.** Let $f(x) = x^2$ on $[0, b]$. Take the uniform partition

$$t_k = \frac{kb}{n}, \quad k = 0, 1, \ldots, n.$$

Then

$$U(f, P) = \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \to \frac{b^3}{3},$$
$$L(f, P) = \frac{b^3}{n^3} \cdot \frac{(n-1)n(2n-1)}{6} \to \frac{b^3}{3}.$$

Hence $L(f) = U(f) = \frac{b^3}{3}$, so

$$\int_0^b x^2 \, dx = \frac{b^3}{3}.$$

**Example 1.6** (Dirichlet Function)**.** The **Dirichlet function** $f : [0, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

This function is bounded but not Riemann integrable. For any partition $P$ of $[0, 1]$, every subinterval contains both rational and irrational numbers. Thus

$$M(f, [t_{k-1}, t_k]) = 1, \quad m(f, [t_{k-1}, t_k]) = 0.$$

Therefore

$$U(f, P) = \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1}) = 1, \qquad L(f, P) = \sum_{k=1}^{n} 0 \cdot (t_k - t_{k-1}) = 0.$$

Hence $U(f) = 1$ and $L(f) = 0$. Since $U(f) \neq L(f)$, the Dirichlet function is not Darboux (and therefore not Riemann) integrable.

# 2 Properties of Darboux Sums

**Lemma 2.1** (Refinement Lemma)**.** If $Q$ is a refinement of $P$ (i.e., $P \subseteq Q$), then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

*Proof.* Adding a point to a partition splits one subinterval into two. The infimum over the union is at least the infimum over each part, so the lower sum does not decrease. Similarly, the upper sum does not increase. □

**Lemma 2.2.** For any two partitions $P, Q$,

$$L(f, P) \leq U(f, Q).$$

*Proof.* The common refinement $P \cup Q$ satisfies

$$L(f, P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f, Q).$$

□

**Theorem 2.3.** For any bounded $f$,

$$L(f) \le U(f).$$

*Proof.* Fix a partition $Q$. For every partition $P$,

$$L(f, P) \le U(f, Q).$$

Taking supremum over $P$ gives $L(f) \le U(f, Q)$. Now taking infimum over $Q$ yields $L(f) \le U(f)$. □

# 3 Criterion for Integrability

**Theorem 3.1** (Darboux Criterion)**.** A bounded function $f$ on $[a, b]$ is Darboux integrable if and only if for every $\varepsilon > 0$ there exists a partition $P$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

*Proof.* ($\Rightarrow$) Suppose $f$ is integrable. Choose partitions $P_1, P_2$ with

$$L(f, P_1) > L(f) - \frac{\varepsilon}{2}, \qquad U(f, P_2) < U(f) + \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$. Then

$$U(f,P)-L(f,P) \le U(f,P_2)-L(f,P_1) < \big(U(f)+\frac{\varepsilon}{2}\big)-\big(L(f)-\frac{\varepsilon}{2}\big) = U(f)-L(f)+\varepsilon = \varepsilon.$$

($\Leftarrow$) Given $\varepsilon > 0$, choose $P$ with $U(f, P) - L(f, P) < \varepsilon$. Then

$$U(f) \le U(f, P) = U(f, P) - L(f, P) + L(f, P) < \varepsilon + L(f, P) \le \varepsilon + L(f).$$

Since $\varepsilon$ is arbitrary, $U(f) \le L(f)$. Together with $L(f) \le U(f)$ we obtain $L(f) = U(f)$. □

# 4 Mesh of a Partition

**Definition 4.1** (Mesh)**.** For a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, the **mesh** of $P$ is

$$\mathrm{mesh}(P) = \max\{t_k - t_{k-1} : k = 1, \ldots, n\}.$$

**Theorem 4.2** (Mesh Criterion)**.** A bounded function $f$ is Darboux integrable if and only if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \mathrm{mesh}(P) < \delta \implies U(f, P) - L(f, P) < \varepsilon.$$

*Proof.* (⇐) Follows immediately from the Darboux criterion.

(⇒) Given $\varepsilon > 0$, choose a partition $P_0 = \{a = u_0 < u_1 < \cdots < u_m = b\}$ with

$$U(f, P_0) - L(f, P_0) < \frac{\varepsilon}{2}.$$

Since $f$ is bounded, there exists $B > 0$ such that $|f(x)| \leq B$ for all $x$. Set $\delta = \frac{\varepsilon}{8mB}$. Let $P$ be any partition with $\text{mesh}(P) < \delta$.

Consider the common refinement $Q = P \cup P_0$. The contribution to the difference $U(f, P) - L(f, P)$ from subintervals of $P$ that contain a point of $P_0$ is at most $2B \cdot \delta \cdot m$. Hence

$$U(f, P) \leq U(f, P_0) + 2B \cdot \delta \cdot m = U(f, P_0) + \frac{\varepsilon}{4},$$

$$L(f, P) \geq L(f, P_0) - 2B \cdot \delta \cdot m = L(f, P_0) - \frac{\varepsilon}{4}.$$

Therefore

$$U(f,P) - L(f,P) \leq \big(U(f,P_0) + \frac{\varepsilon}{4}\big) - \big(L(f,P_0) - \frac{\varepsilon}{4}\big) = \big(U(f,P_0) - L(f,P_0)\big) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

# 5 Riemann Sums

**Definition 5.1** (Riemann Sum)**.** For a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ and chosen points $x_k \in [t_{k-1}, t_k]$, the corresponding **Riemann sum** is

$$S(f, P, \{x_k\}) = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}).$$

**Definition 5.2** (Riemann Integrability)**.** A bounded function $f$ is **Riemann integrable** if there exists a number $r$ such that

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \text{mesh}(P) < \delta \implies |S(f, P, \{x_k\}) - r| < \varepsilon$$

for every choice of the intermediate points $x_k$. The number $r$ is then called the **Riemann integral** of $f$.

**Theorem 5.3** (Equivalence)**.** A bounded function $f$ on $[a, b]$ is Darboux integrable if and only if it is Riemann integrable, and the two integrals coincide.

*Proof.* (⇒) Assume $f$ is Darboux integrable with integral $I = L(f) = U(f)$. Given $\varepsilon > 0$, choose $\delta > 0$ by the mesh criterion so that $\text{mesh}(P) < \delta$ implies $U(f, P) - L(f, P) < \varepsilon$. For any Riemann sum $S$ corresponding to such a partition,

$$L(f, P) \leq S \leq U(f, P) \quad \text{and} \quad L(f, P) \leq I \leq U(f, P).$$

Hence $|S - I| \leq U(f, P) - L(f, P) < \varepsilon$.

($\Leftarrow$) Suppose $f$ is Riemann integrable with integral $r$. Given $\varepsilon > 0$, choose $\delta > 0$ so that for every partition $P$ with $\mathrm{mesh}(P) < \delta$ and any choice of intermediate points, $|S - r| < \varepsilon$. For such a partition, pick $x_k$ with $f(x_k) < m(f, [t_{k-1}, t_k]) + \frac{\varepsilon}{b-a}$. Then

$$S < L(f,P) + \frac{\varepsilon}{b-a}(b-a) = L(f,P) + \varepsilon.$$

Thus $r < L(f,P) + \varepsilon$, and taking supremum over all such partitions gives $r \leq L(f) + \varepsilon$. Since $\varepsilon$ is arbitrary, $r \leq L(f)$. Similarly, $r \geq U(f)$. But $L(f) \leq U(f)$, so $L(f) = U(f) = r$, and $f$ is Darboux integrable. $\square$