Properties of the Riemann Integral

1 Integrability of Monotone and Continuous Functions

Theorem 1.1 (Monotone functions are integrable)**.** If f is monotone (increasing or decreasing) on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Proof. Assume f is increasing (the decreasing case is similar). Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition. Because f is increasing,

$$M(f, [t_{k-1}, t_k]) = f(t_k), \quad m(f, [t_{k-1}, t_k]) = f(t_{k-1}).$$

Hence

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} \big(f(t_k) - f(t_{k-1})\big)(t_k - t_{k-1}).$$

Since $t_k - t_{k-1} \le \text{mesh}(P)$, we have

$$U(f, P) - L(f, P) \le \text{mesh}(P) \sum_{k=1}^{n} \big(f(t_k) - f(t_{k-1})\big) = \text{mesh}(P)\,\big(f(b) - f(a)\big).$$

Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{f(b) - f(a) + 1}$. If $\text{mesh}(P) < \delta$, then

$$U(f, P) - L(f, P) < \varepsilon.$$

By the Darboux criterion, f is integrable. □

Theorem 1.2 (Continuous functions are integrable)**.** If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Proof. Since $[a, b]$ is compact, f is uniformly continuous. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Let P be any partition with $\text{mesh}(P) < \delta$. On each subinterval $[t_{k-1}, t_k]$,

$$M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) < \frac{\varepsilon}{b - a}.$$

Therefore

$$U(f,P)-L(f,P) = \sum_{k=1}^{n}\big(M(f,[t_{k-1},t_k])-m(f,[t_{k-1},t_k])\big)(t_k-t_{k-1}) < \frac{\varepsilon}{b-a}\sum_{k=1}^{n}(t_k-t_{k-1}) = \varepsilon.$$

Thus f is integrable. □

2 Linearity and Order Properties

Theorem 2.1 (Linearity of the integral)**.** If f and g are integrable on $[a,b]$ and $c \in \mathbb{R}$, then

(i) cf is integrable and $\displaystyle\int_a^b cf = c\int_a^b f$.

(ii) $f+g$ is integrable and $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Proof. Both statements follow easily from the corresponding properties of Riemann sums. For any partition P and choice of intermediate points,

$$S(cf,P) = c\,S(f,P), \qquad S(f+g,P) = S(f,P)+S(g,P).$$

Taking limits as $\mathrm{mesh}(P)\to 0$ gives the desired formulas. □

Theorem 2.2 (Order preservation)**.** If f and g are integrable on $[a,b]$ and $f(x)\le g(x)$ for all $x\in[a,b]$, then

$$\int_a^b f \le \int_a^b g.$$

Proof. The function $g-f$ is integrable (by linearity) and nonnegative. Hence every Riemann sum for $g-f$ is nonnegative, and so is its limit:

$$\int_a^b (g-f) \ge 0.$$

Using linearity again, $\int_a^b g - \int_a^b f \ge 0$. □

Corollary 2.3 (Integral of a nonnegative continuous function)**.** If g is continuous, nonnegative on $[a,b]$, and $\displaystyle\int_a^b g = 0$, then g is identically zero on $[a,b]$.

Proof. Suppose, for contradiction, that $g(x_0) > 0$ for some $x_0 \in [a,b]$. By continuity, there exists an interval $[c,d]\subseteq[a,b]$ containing x_0 such that $g(x)\ge \alpha > 0$ on $[c,d]$. Then

$$\int_a^b g \ge \int_c^d g \ge \alpha\,(d-c) > 0,$$

contradicting the hypothesis that the integral is zero. □

3 Absolute Value and Additivity

Theorem 3.1 (Integrability of $|f|$)**.** If f is integrable on $[a, b]$, then $|f|$ is also integrable and

$$\left| \int_a^b f \right| \le \int_a^b |f|.$$

Proof. For any subinterval $I \subseteq [a, b]$,

$$M(|f|, I) - m(|f|, I) \le M(f, I) - m(f, I),$$

because the oscillation of $|f|$ does not exceed that of f. Hence

$$U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P).$$

Since f is integrable, the right-hand side can be made arbitrarily small, so $|f|$ is integrable.

The inequality $-|f| \le f \le |f|$ together with order preservation gives

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|,$$

which is equivalent to $\left| \int_a^b f \right| \le \int_a^b |f|.$ □

Theorem 3.2 (Additivity over intervals)**.** If f is integrable on $[a, b]$ and $a < c < b$, then f is integrable on $[a, c]$ and $[c, b]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Conversely, if f is integrable on $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ and the same equality holds.

Proof. Assume first that f is integrable on $[a, b]$. Given $\varepsilon > 0$, choose a partition P of $[a, b]$ with $U(f, P) - L(f, P) < \varepsilon$. Adding the point c if necessary, we obtain a refinement P' that splits into a partition P_1 of $[a, c]$ and a partition P_2 of $[c, b]$. Then

$$U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) \le U(f, P') - L(f, P') \le U(f, P) - L(f, P) < \varepsilon.$$

Thus both $U(f, P_i) - L(f, P_i)$ are arbitrarily small, so f is integrable on each subinterval. Moreover,

$$L(f, P_1) + L(f, P_2) \le L(f, P') \le \int_a^b f \le U(f, P') \le U(f, P_1) + U(f, P_2).$$

Taking suprema of lower sums and infima of upper sums gives

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

The converse direction is proved similarly by combining partitions of $[a, c]$ and $[c, b]$ into a partition of $[a, b]$. □