4: Completeness Axiom

1 Bounded Sets and Extremal Elements

Definition 1.1. Let $\emptyset \neq S \subseteq \mathbb{R}$.

1. If S contains a largest element s_0 (i.e., $s \leq s_0$ for all $s \in S$), then s_0 is called the *maximum* of S, denoted $\max S$.
2. If S contains a smallest element, then it is called the *minimum* of S, denoted $\min S$.

Example 1.2. If S is finite, then it always has both maximum and minimum. For example:

- $\max\{-1, 3, 2, 0\} = 3$
- $\min\{-1, 3, 2, 0\} = -1$

Definition 1.3. For $a, b \in \mathbb{R}$ with $a \leq b$:

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed interval)
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ (open interval)

Example 1.4.
- $\max[a, b] = b$, $\min[a, b] = a$
- (a, b) has no maximum or minimum
- $\mathbb{Z}$ and $\mathbb{Q}$ have no minimum or maximum
- $\min \mathbb{N} = 1$
- $S = \{r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2}\}$ has no maximum since $\sqrt{2} \notin S$
- $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ has no minimum since $0 \notin S$

2 Boundedness

Definition 2.1. Let $\emptyset \neq S \subseteq \mathbb{R}$.

1. If there exists $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$, then M is called an *upper bound* of S and S is said to be *bounded above*.
2. If there exists $m \in \mathbb{R}$ such that $m \leq s$ for all $s \in S$, then m is called a *lower bound* of S and S is said to be *bounded below*.
3. S is *bounded* if it is bounded both above and below, i.e., there exist $m, M \in \mathbb{R}$ such that $S \subseteq [m, M]$.

Example 2.2.
- If $\max S$ exists, then it is an upper bound of S
- If $\min S$ exists, then it is a lower bound of S
- For $(3, 5)$, both 5 and 6 are upper bounds
- $\mathbb{Z}, \mathbb{Q}, \mathbb{N}, \mathbb{R}$ are not bounded above
- $S = \{r \in \mathbb{Q} : r \leq \sqrt{2}\}$ has no maximum but is bounded above (e.g., by 2 and $\sqrt{2}$)
- $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ has 0 as its greatest lower bound

3 Supremum and Infimum

Definition 3.1. Let $\emptyset \neq S \subseteq \mathbb{R}$.

1. If S is bounded above and M is the least upper bound, then M is called the *supremum* of S, denoted $\sup S$.
2. If S is bounded below and m is the greatest lower bound, then m is called the *infimum* of S, denoted $\inf S$.

Theorem 3.2 (Characterization of Supremum)**.** *If S is bounded above, then $M = \sup S$ if and only if:*

1. *$s \leq M$ for all $s \in S$*
2. *For any $M_1 < M$, there exists $s \in S$ such that $s > M_1$*

Example 3.3. If S has a maximum, then $\max S = \sup S$. For example:

$$\sup\{r \in \mathbb{Q} : r \leq \sqrt{2}\} = \sqrt{2}$$

4 Completeness Axiom

Axiom 1 (Completeness Axiom)**.** Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

Theorem 4.1. $\mathbb{Q}$ *is not complete.*

Proof. Consider $S = \{r \in \mathbb{Q} : r \leq \sqrt{2}\} \subseteq \mathbb{Q}$. Then $\sup S = \sqrt{2} \notin \mathbb{Q}$. Note that S can be defined purely using $\mathbb{Q}$ as $S = \{r \in \mathbb{Q} : r^2 \leq 2\} \cup \{r \in \mathbb{Q} : r < 0\}$. □

Theorem 4.2. *Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound* $\inf S$.

Proof. Suppose S is bounded below by m. Define $-S = \{-s : s \in S\}$. Then $-S$ is bounded above by $-m$. By the completeness axiom, $-S$ has a least upper bound M. Then $-M = \inf S$. □

Homework Problems

4.2, 4.6, 4.10, 4.12, 4.14