5. Archimedean Property

1 Archimedean Property

Theorem 1.1 (Archimedean Property of $\mathbb{R}$)**.** *If $a > 0$ and $b > 0$, then there exists a positive integer n such that $na > b$.*

Proof. Suppose, for contradiction, that there exist $a > 0$ and $b > 0$ such that $na \leq b$ for all $n \in \mathbb{N}$. Then b is an upper bound for the set $S = \{na : n \in \mathbb{N}\}$.

By the completeness axiom, $\sup S$ exists. Denote $S_0 = \sup S$. Since $a > 0$, we have $S_0 - a < S_0$. As S_0 is the least upper bound, there exists $n_0 a \in S$ such that $n_0 a > S_0 - a$. But then $(n_0 + 1)a > S_0$, which contradicts that S_0 is an upper bound for S. Therefore, the assumption is false and the theorem holds. □

Corollary 1.2. *If $a > 0$, then there exists $n \in \mathbb{N}$ such that $a > \frac{1}{n}$.*

Proof. Apply the Archimedean property with $b = 1$ to get $na > 1$, so $a > \frac{1}{n}$. □

Corollary 1.3. *If $b > 0$, then there exists $n \in \mathbb{N}$ such that $b < n$.*

Proof. Apply the Archimedean property with $a = 1$ to get $n > b$. □

2 Denseness of Rational Numbers

Theorem 2.1 (Denseness of $\mathbb{Q}$ in $\mathbb{R}$)**.** *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. We want to find $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a < \frac{m}{n} < b$, or equivalently, $na < m < nb$.

Since $b - a > 0$, by the Archimedean property, there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$, so $nb - na > 1$.

By the Archimedean property again, there exists $k \in \mathbb{N}$ such that $k > \max\{|na|, |nb|\}$, so $-k < na < nb < k$.

Let $K = \{j \in \mathbb{Z} : -k \leq j \leq k\}$ and consider the set $\{j \in K : na < j\}$. This set is finite and nonempty (it contains k). Let $m = \min\{j \in K : na < j\}$.

Since $m > -k$, we have $m - 1 \in K$. By the minimality of m, we have $m - 1 \leq na$. Therefore:

$$na < m \leq na + 1 < nb$$

So $na < m < nb$, which implies $a < \frac{m}{n} < b$. □

Homework Problems

5.1, 5.2, 5.3, 5.4, 5.5, 5.6