6. Dedekind Cuts

1 Notation of infinity

Definition 1.1. We define extended intervals:

- $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$
- $(a, \infty) = \{x \in \mathbb{R} : x > a\}$
- $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$
- $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$

The symbols ∞ and $-\infty$ represent positive and negative infinity, respectively. They are not considered as real numbers.

Definition 1.2.

- (a, b), (a, ∞), $(-\infty, a)$ are called *open intervals*
- $[a, b]$, $[a, \infty)$, $(-\infty, a]$ are called *closed intervals*
- If S is not bounded above, we write $\sup S = \infty$
- If S is not bounded below, we write $\inf S = -\infty$

2 Dedekind Cuts

Definition 2.1. A *Dedekind cut* is a subset $\alpha \subset \mathbb{Q}$ ($\alpha \neq \emptyset$, $\alpha \neq \mathbb{Q}$) satisfying:

1. $\alpha \neq \emptyset$
2. If $r \in \alpha$, $s \in \mathbb{Q}$, and $s < r$, then $s \in \alpha$
3. α contains no largest rational number

Example 2.2.

- $\alpha = \{r \in \mathbb{Q} : r < \frac{1}{3}\}$ is a Dedekind cut
- $\alpha = \{r \in \mathbb{Q} : r < \sqrt{2}\}$ is a Dedekind cut

Theorem 2.3. *Given any real number a, the set $\{r \in \mathbb{Q} : r < a\}$ is a Dedekind cut.*

Theorem 2.4. *Given any Dedekind cut α, the set α is bounded above. By the completeness axiom, $\sup \alpha \in \mathbb{R}$ exists.*

3 Construction of Real Numbers

Theorem 3.1. *The real numbers can be constructed using Dedekind cuts. Specifically:*

1. *Each Dedekind cut corresponds to a real number*
2. *Addition is defined as $\alpha + \beta = \{r_1 + r_2 : r_1 \in \alpha, r_2 \in \beta\}$*
3. *Order is defined as $\alpha \leq \beta$ if for all $r_1 \in \alpha$, there exists $r_2 \in \beta$ such that $r_1 \leq r_2$*

Example 3.2. If $\alpha = \{r \in \mathbb{Q} : r < 2\}$ and $\beta = \{r \in \mathbb{Q} : r < 3\}$, then we expect $\alpha \cdot \beta = \{r \in \mathbb{Q} : r < 6\}$. However, multiplication requires careful definition to handle negative numbers properly.

Theorem 3.3. *When $\mathbb{R}$ is constructed using Dedekind cuts, the completeness axiom becomes a theorem rather than an axiom.*