Limits of Sequences

Definition 1 (Sequence)**.** *A **sequence** (of real numbers) is a function* $\alpha : \mathbb{N} \to \mathbb{R}$*. It is denoted by*

$$a(1), a(2), a(3), \ldots \quad \textit{or} \quad a_1, a_2, a_3, \ldots$$

Example 1. *The sequence defined by* $S_n = \frac{1}{n^2}$ *is*

$$\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots$$

Definition 2 (Limit of a Sequence)**.** *A sequence* $(S_n)_{n\in\mathbb{N}}$ *of real numbers **converges** to a real number* S *if for every* $\epsilon > 0$*, there exists a number* $N \in \mathbb{N}$ *such that for all* $n > N$*,*

$$|S_n - S| < \epsilon.$$

We denote this by

$$\lim_{n\to\infty} S_n = S, \quad \textit{or} \quad S_n \to S \textit{ as } n \to \infty.$$

The number S *is called the **limit** of the sequence* (S_n)*. A sequence that converges is called a **convergent sequence**; otherwise, it is called a **divergent sequence**.*

Example 2. $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof. For any $\epsilon > 0$, choose $N = \frac{1}{\epsilon}$. Then for all $n > N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \epsilon.$$

Hence, by definition, the limit is 0. □

Lemma 1 (Uniqueness of the Limit)**.** *The limit of a convergent sequence is unique. That is, if* $\lim_{n\to\infty} S_n = S$ *and* $\lim_{n\to\infty} S_n = S'$*, then* $S = S'$*.*

Proof. For any $\epsilon > 0$, there exist N_1, N_2 such that:

$$|S_n - S| < \frac{\epsilon}{2} \quad \text{for all } n > N_1,$$
$$|S_n - S'| < \frac{\epsilon}{2} \quad \text{for all } n > N_2.$$

Let $N = \max\{N_1, N_2\}$. Then for all $n > N$,

$$|S - S'| = |S - S_n + S_n - S'| \le |S - S_n| + |S_n - S'| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since ϵ was arbitrary, it must be that $S = S'$. □

Example 3. *The sequence* $a_n = (-1)^n$ *is divergent.*

Proof. Suppose, for contradiction, that $\lim_{n\to\infty}(-1)^n = a$ for some $a \in \mathbb{R}$. Choose $\epsilon = \frac{1}{3}$. Then there exists an N such that for all $n > N$,

$$|(-1)^n - a| < \frac{1}{3}. \tag{*}$$

Now, consider two consecutive terms, n and $n+1$, both greater than N:

$$\begin{aligned}
|(-1)^{n+1} - a| &= |-(-1)^n - a| \\
&= |(-1)^{n+1} - (-1)^n + (-1)^n - a| \\
&\geq \left|(-1)^{n+1} - (-1)^n\right| - |(-1)^n - a| \quad \text{(reverse triangle inequality)} \\
&= |-2(-1)^n| - |(-1)^n - a| \\
&= 2 - |(-1)^n - a| \\
&> 2 - \frac{1}{3} = \frac{5}{3}.
\end{aligned}$$

This contradicts (*), which requires $|(-1)^{n+1} - a| < \frac{1}{3}$. Therefore, the limit cannot exist. □