Some proofs

Definition 1 (Bounded Sequence)**.** *A sequence* (S_n) *is* ***bounded*** *if there exists a real number* M *such that* $|S_n| \leq M$ *for all* $n \in \mathbb{N}$*.*

Theorem 1 (Convergence Implies Boundedness)**.** *Every convergent sequence is bounded.*

Proof. Suppose $\lim_{n\to\infty} S_n = S$. For $\epsilon = 1$, there exists an N such that for all $n > N$,

$$|S_n - S| < 1.$$

By the triangle inequality,

$$|S_n| = |S_n - S + S| \leq |S_n - S| + |S| < 1 + |S|.$$

Now, let

$$M = \max\{|S_1|, |S_2|, \ldots, |S_N|, 1 + |S|\}.$$

Then $|S_n| \leq M$ for all $n \in \mathbb{N}$, so the sequence is bounded. □

Theorem 2 (Algebraic Limit Theorems)**.** *Suppose* $\lim_{n\to\infty} s_n = s$ *and* $\lim_{n\to\infty} t_n = t$*. Then:*

1. ***Constant Multiple:*** $\lim_{n\to\infty}(k \cdot s_n) = k \cdot s$ *for any* $k \in \mathbb{R}$*.*
2. ***Sum:*** $\lim_{n\to\infty}(s_n + t_n) = s + t$*.*
3. ***Product:*** $\lim_{n\to\infty}(s_n \cdot t_n) = s \cdot t$*.*
4. ***Quotient:*** *If* $s_n \neq 0$ *for all* n *and* $s \neq 0$*, then* $\lim_{n\to\infty}\left(\frac{t_n}{s_n}\right) = \frac{t}{s}$*.*

Proof. (Proof sketches)

1. For $k \neq 0$, given $\epsilon > 0$, find N such that $|s_n - s| < \frac{\epsilon}{|k|}$ for $n > N$. Then $|ks_n - ks| = |k||s_n - s| < \epsilon$. The case $k = 0$ is trivial.
2. Use the triangle inequality: $|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t|$. Choose N large enough so both terms are less than $\epsilon/2$.
3. Use the identity:

   $$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \leq |s_n||t_n - t| + |t||s_n - s|.$$

   Since (s_n) is bounded ($|s_n| < M$), choose N such that $|s_n - s| < \frac{\epsilon}{2(|t|+1)}$ and $|t_n - t| < \frac{\epsilon}{2M}$.
4. First prove the lemma for reciprocals (below). The main result follows by applying the product rule to $t_n \cdot \frac{1}{s_n}$.

□

Lemma 3 (Limit of the Reciprocal)**.** *If* $\lim_{n\to\infty} S_n = S$*,* $S_n \neq 0$ *for all* n*, and* $S \neq 0$*, then* $\lim_{n\to\infty} \frac{1}{S_n} = \frac{1}{S}$*.*

Proof. Since $S_n \to S \neq 0$, one can show there exists $m > 0$ such that $|S_n| > m$ for all n. For any $\epsilon > 0$, there exists N such that for all $n > N$,

$$|S_n - S| < \epsilon \cdot m|S|.$$

Then,

$$\left|\frac{1}{S_n} - \frac{1}{S}\right| = \frac{|S - S_n|}{|S_n||S|} \leq \frac{|S - S_n|}{m|S|} < \frac{\epsilon \cdot m|S|}{m|S|} = \epsilon.$$

Hence, $\frac{1}{S_n} \to \frac{1}{S}$. □