# Limit Theorems

**Theorem 1** (Important Special Limits)**.** *The following limits hold:*

1. *For any* $p > 0$, $\lim_{n\to\infty} \frac{1}{n^p} = 0$.
2. *If* $|a| < 1$, *then* $\lim_{n\to\infty} a^n = 0$.
3. $\lim_{n\to\infty} n^{1/n} = 1$.
4. *For any* $a > 0$, $\lim_{n\to\infty} a^{1/n} = 1$.

*Proof.* (Proof sketches)

1. For $\epsilon > 0$, choose $N = \epsilon^{-1/p}$. Then for $n > N$, $\left|\frac{1}{n^p}\right| < \frac{1}{N^p} = \epsilon$.
2. Write $|a| = \frac{1}{1+b}$ for some $b > 0$. Then $|a^n| = \frac{1}{(1+b)^n}$. By Bernoulli's inequality, $(1+b)^n \geq 1+nb > nb$, so $|a^n| < \frac{1}{nb}$. For $\epsilon > 0$, choose $N > \frac{1}{\epsilon b}$.
3. Let $S_n = n^{1/n} - 1$, so $n = (1 + S_n)^n$. Using the Binomial Theorem for $n \geq 2$:
$$n > \binom{n}{2} S_n^2 = \frac{n(n-1)}{2} S_n^2.$$
Solving gives $S_n^2 < \frac{2}{n-1}$, so $0 \leq S_n < \sqrt{\frac{2}{n-1}}$. By the Sandwich Theorem and (1), $S_n \to 0$, so $n^{1/n} \to 1$.
4. If $a > 1$, then for large $n$, $1 \leq a^{1/n} \leq n^{1/n}$. Apply the Sandwich Theorem using (3). If $0 < a < 1$, then $\frac{1}{a} > 1$, and $a^{1/n} = \frac{1}{(1/a)^{1/n}} \to \frac{1}{1} = 1$.

□

**Definition 1** (Divergence to Infinity)**.** *We write* $\lim_{n\to\infty} s_n = \infty$ *if for every real number* $M$*, there exists a number* $N$ *such that* $s_n > M$ *for all* $n > N$.

**Example 1.** $\lim_{n\to\infty} n^2 = \infty$.

**Theorem 2** (Operations with Infinite Limits)**.** *Suppose* $\lim_{n\to\infty} S_n = \infty$.

1. *If* $\lim_{n\to\infty} t_n = t > 0$ *(a positive finite limit) or* $\lim_{n\to\infty} t_n = \infty$, *then* $\lim_{n\to\infty}(S_n t_n) = \infty$.
2. *If* $S_n > 0$ *for all* $n$, *then* $\lim_{n\to\infty} S_n = \infty$ *if and only if* $\lim_{n\to\infty} \frac{1}{S_n} = 0$.

*Proof.* (Proof sketches)

1. Since $t_n$ is eventually positive and bounded away from zero (or tends to infinity), we can find a positive lower bound $m$ for $t_n$. For any $M > 0$, since $S_n \to \infty$, we can find $N$ such that $S_n > M/m$ for $n > N$. Then $S_n t_n > (M/m) \cdot m = M$.

2. ($\Rightarrow$): Given $\epsilon > 0$, let $M = 1/\epsilon$. Since $S_n \to \infty$, there exists $N$ such that for all $n > N$, $S_n > M$. Then $\left|\frac{1}{S_n} - 0\right| = \frac{1}{S_n} < \frac{1}{M} = \epsilon$.
($\Leftarrow$): The converse is similar and is left as an exercise.

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